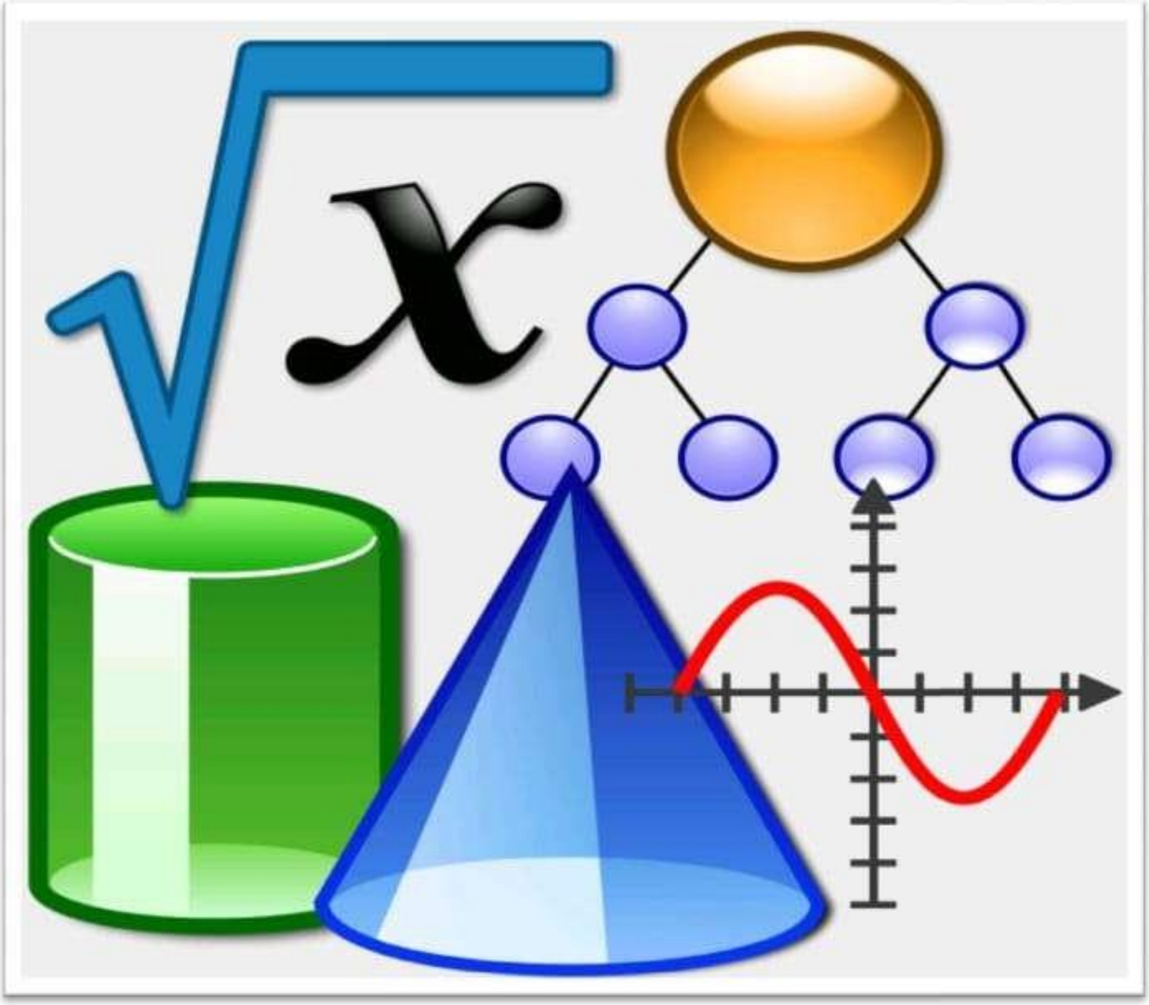
DEPARTMENT OF BASIC SCIENCE
ENGINEERING MATHEMATICS-III



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CONTENTS

CHAPTER - 1

DETERMINANT

INTRODUCTION :

The study of determinants was started by Leibnitz in the concluding portion of seventeenth century. This was latter developed by many mathematician like Cramer, Lagrange, Laplace, Cauchy, Jocobi. Now the determinants are used to study some of aspects of matrices.

Determinant : If the linear equations

$a_1x + b_1 = 0$

and $a_2x + b_2 = 0$

have the same solution, then $\frac{b_1}{a_1} = \frac{b_2}{a_2}$

or $a_1b_2 - a_2b_1 = 0$

The expression $(a_1b_2 - a_2b_1)$ is called a **determinant** and is denoted by symbol.

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ or by (a_1b_2) where a_1, a_2, b_1 & b_2 are called the elements of the **determinant**. The elements in the horizontal direction from rows, and those in the vertical direction form **columns**. The determinant

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has two rows and two coloums. So it is called a **determinant of the second order** and it has 2! = 2 terms in its expansion of which one is positive and other is negative. The diagonal term, or the leading term of the determinant is a_1b_2 whose sign is positive.

Again if the linear equations

$a_1x + b_1y + c_1 = 0$ .... (i)

$a_2x + b_2y + c_2 = 0$ .... (ii)

$a_3x + b_3y + c_3 = 0$ .... (iii)

have the same solutions, we have from the last two equations by cross-multiplication.

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{1}{a_2b_3 - a_3b_2}$$

or $x = \frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}$, $y = \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}$

These values of x and y must satisfy the first equation. Hence $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$

or $a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1$ is denoted by the symbol

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ or by $(a_1b_2c_3)$ and has three rows, and three columns. So it is called a **determinant of the third order** and it has 3! = 6 terms of which three terms are positive, and three terms are negative.

MINORS

Minors : The determinant obtained by suppressing the row and the column in which a particular element occurs is called the minor of that element.

Therefore, in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ and that of c_3 is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and so on.

The minor of any element in a third order determinant is thus a second order determinant.

The minors of $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ are denoted by $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ respectively.

Hence $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $A_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, $A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

$B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, $B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$, $C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, $C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

If D stands for the value of the determinant, then

$D = a_1A_1 - b_1B_1 + c_1C_1 = a_1A_1 - a_2A_2 + a_3A_3$

Cofactors : The cofactor of any element in a determinant is its coefficient in the expansion of the determinant.

It is therefore equal to the corresponding minor with a proper sign.

For calculation of the proper sign to be attached to the minor of the element, one has to consider $(-1)^{i+j}$ and to multiply this sign with the minor of the element a_{ij} where i and j are respecively the row and the column to which the element a_{ij} belongs.

Thus $C_{ij} = (-1)^{i+j} M_{ij}$ Where C_{ij} and M_{ij} are respectively the cofactor and the minor of the element a_{ij}.

The cofactor of any element is generally denoted by the corresponding capital letter.

Thus for the determinant $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, cofactor of a_1 is

$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_1 is $B_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$

that of c_1 is $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

(The sign is $(-1)^{1+3} = 1$), and so on.

We see that minors and cofactors are either equal of differ in sign only.

With this notation the determinant may be expanded in the form,

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1A_1 + b_1B_1 + c_1C_1$

Similarly we express $= a_2A_2 + b_2B_2 + c_2C_2$

$= a_3A_3 + b_3B_3 + c_3C_3$

By expanding with respect to the elements of the first column, we can write

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$= a_1A_1 + a_2A_2 + a_3A_3$ Similarly

$= b_1B_1 + b_2B_2 + b_3B_3$

$= c_1C_1 + c_2C_2 + c_3C_3$

Thus the determinant can be expressed as the sum of the product of the elements of any row (or column) and the corresponding cofactors of the respective elements of the same row (or column).

PROPERTIES OF DETERMINANT

I. The value of a determinant is unchanged if the rows are written as columns and columns as rows.

If the rows and coloums are interchanged in the determinant of 2nd order $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, the determinant becoems $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Each of the two $= a_1b_2 - a_2b_1$

$$\therefore \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

In the third order determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

if the rows and column are interchanged, it

becomes $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$ (say)

If D is expanded by taking the constituents of the first column and D' is expanded by taking the constituents of the first row, then

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\text{and } \Delta' = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

\ D = D' (since the value of determinant of 2nd orders is unchanged if rows and columns are interchanged).

II. If two adjacent rows and columns of the determinant are interchanged the sign of the determinant is changed but its absolute value remains unaltered.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

D' has been obtained by interchanging the first and second rows of D

Expanding each determinant by the constituents of the first column.

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

and $D' = a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix}$

$$= -a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$\left[\text{since } \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} = b_2c_1 - c_2b_1 \text{ and } \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1 \right] = -\Delta$

In this way it can be proved that only the sign changes if any other two adjacent rows or columns are interchanged.

III. If two rows or columns of a determinant are identical, the determinant vanishes.

Let $\Delta_2 = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$

The first two columns in the determinant are identical. If the first and second columns are interchanged, then the resulting determinant becomes $-D_2$ by II. But since these two columns are identical, the determinant remains unaltered by the interchange.

\ $D_2 = -D_2$ or, $2D_2 = 0$

\ $D_2 = 0$

IV. If each constitutent in any row or any column is multiplied by the same factor, then the determinant is multiplied by that factor.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The determinant obtained when the constituents of the first row are multiplied by m is

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1A_1 - ma_2A_2 + ma_3A_3$$

$= m[a_1A_1 - a_2A_2 + a_3A_3] = mD$

V. If each constituent in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two other determinants in the determinant.

In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let $a_1 = t_1 + m_1 + n_1$, $a_2 = t_2 + m_2 + n_2$, $a_3 = t_3 + m_3 + n_3$
Then the given determinant

$$= \begin{vmatrix} t_1+m_1+n_1 & b_1 & c_1 \\ t_2+m_2+n_2 & b_2 & c_2 \\ t_3+m_3+n_3 & b_3 & c_3 \end{vmatrix}$$

$= (t_1 + m_1 + n_1) A_1 - (t_2 + m_2 + n_2) A_2 + (t_3 + m_3 + n_3) A_3$
$= (t_1A_1 - t_2A_2 + t_3A_3) + (m_1A_1 - m_2A_2 + m_3A_3) + (n_1A_1 - n_2A_2 + n_3A_3)$

$$= \begin{vmatrix} t_1 & b_1 & c_1 \\ t_2 & b_2 & c_2 \\ t_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} n_1 & b_1 & c_1 \\ n_2 & b_2 & c_2 \\ n_3 & b_3 & c_3 \end{vmatrix}$$

It can be similarly proved that

$$\begin{vmatrix} a_1+p_1 & b_1+q_1 & c_1 \\ a_2+p_2 & b_2+q_2 & c_2 \\ a_3+p_3 & b_3+q_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & q_1 & c_1 \\ a_2 & q_2 & c_2 \\ a_3 & q_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ p_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 & c_1 \\ p_2 & q_2 & c_2 \\ p_3 & q_3 & c_3 \end{vmatrix}$$

VI. If the constituents of any row (or column) be increased or decreased by any equimultiples of the corresponding constituents of one or more of the other rows (or columns) the value of the determinant remains unaltered.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The determinant obtained, when the constituents of first column are increased by *l* times the second column *m* times the corresponding constituents of the third column is

$$\begin{vmatrix} a_1+lb_1+mc_1 & b_1 & c_1 \\ a_2+lb_2+mc_2 & b_2 & c_2 \\ a_3+lb_3+mc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} lb_1 & b_1 & c_1 \\ lb_2 & b_2 & c_2 \\ lb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mc_1 & b_1 & c_1 \\ mc_2 & b_2 & c_2 \\ mc_3 & b_3 & c_3 \end{vmatrix} \text{ (by v)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} \text{ (by iv)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS Cramer's Rule :

A method is given below for solving three simultaneous linear equations in three unknowns. This method may also be applied to solve 'n' equations in 'n' unknowns.
Consider the system of equations.

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \text{.........(1)}$$

Where the coefficients are real.

The coefficient of x, y, z as noted in equations (1) may be used to form the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which is called the determinant of the system.

If $\Delta \neq 0$, the solution of (1) is given by $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$, where Δ_r ; r = 1, 2 , 3 is the determinant obtained from Δ by replacing the r^{th} column by d_1, d_2, d_3.

Example –1 : Find the value of $\begin{vmatrix} 5 & -2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix}$

Solution : The value of the given determinant

$$= 5\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 8 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 8 & 1 \end{vmatrix}$$

$= 5(0–2)–2(9–16)+1(3–0)$

$= –10+14+3=7$

Example – 2. **Prove that** $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$ **= abc (a – b) (b – c) (c – a)**

Solution : L.H.S. $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (Taking a, b, c, from R_1, R_2, R_3)

$= abc \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$, replacing R_1 by $R_1 - R_2$ and R_2 by $R_2 - R_3$)

$= abc\,(a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$ (Taking (a – b) & (b – c) common from R_1 & R_2 respectively)

$$= abc\,(a-b)\,(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix} = abc\,(a-b)\,(b-c)\,(c-a)$$

Assignment

1. Find minors & cofactors of the determinants $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

2. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

3. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

r] r

MATRIX

MATRIX AND ITS ORDER

INTRODUCTION :

In modern engineering mathematics matrix theory is used in various areas. It has special relationship with systems of linear equations which occour in many engineering processes.

A matrix is a reactangular array of numbers arranged in rows (horizontal lines) and columns (verti-cal lines). If there are 'm' rows and 'n' Column's in a matrix, it is called an 'm' by 'n' matrix or a matrix of order m × n. The first letter in mxn denotes the number of rows and the second letter 'n' denotes the number of columns. Generally the capital letters of the alphabet are used to denote matrices and the actual matrix is enclosed in parantheses.

$$\text{Hence } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & -- & a_{1n} \\ a_{21} & a_{22} & a_{23} & -- & a_{2n} \\ a_{31} & a_{32} & a_{33} & -- & a_{3n} \\ & & & & \\ a_{m1} & a_{m2} & a_{m3} & -- & a_{mn} \end{bmatrix}$$

is a matrix of order m × n and 'a' $_{ij}$ denotes the element in the ith row and jth column. For example a_{23} is the element in the 2nd row and third column. Thus the matrix 'A' may be written as (a_{ij}) where i takes values from 1 to m to represent row and j takes values from 1 to n to represent column.

If m = n, the matrix A is called a square matrix of order n × n (or simply n). Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & ---- & a_{nn} \end{bmatrix}$$

is a square matrix of order n. The determinant of order n, $\begin{vmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & ---- & a_{nn} \end{vmatrix}$

which is associated with the matrix 'A' is called the determinant of the matrix and is denoted by det A or |A|.

TYPES OF MATRICES WITH EXAMPLES

(a) **Row Matrix :** A matrix of order 1 × n is called a row matrix. For example (1 2), (a b c) are row matrices of order 1 × 2 and 1 × 3 respectively.

(b) Column Matrix : A matrix of order m × 1 is called a column matrix. The matrices $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}a\\b\end{bmatrix}$ are column matrices of order 3 × 1 and 2 × 1 respectively.

(c) Zero matrix : If all the elements of a matrix are zero it is called the zero matrix, (or null matrix) denoted by (0). The zero matrix may be of any order. Thus (0), (0, 0), $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ are all zero matrices.

(d) Unit Matrix : The square matrix whose elements on its main diagonal (left top to right bottom) are 1's and rest of its elements are 0's is called unit matrix. It is denoted by I and it may be of any order. Thus (1) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ are unit matrices of order 1, 2, 3 respectively.

(e) Singular and non -singular matrices : A square matrix A is said to be singular if and only if its determinant is zero and is said to be non-singular (or regular) if det A ≠ 0.

For example $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$ is a non singular matrix.

For $\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix} = 4-6 = -2 \neq 0$ and $\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}$ is a singular matrix

i.e. $\begin{vmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{vmatrix} = 0$

Adjoint of a Matrix :

The adjoint of a matrix A is the transpose of the matrix obtained replacing each element a_{ij} in A by its cofactor A_{ij}. The adjoint of A is written as adj A. Thus if

$$A = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix} \text{ then adj } A = \begin{pmatrix}A_{11} & A_{21} & A_{31}\\A_{12} & A_{22} & A_{32}\\A_{13} & A_{23} & A_{33}\end{pmatrix}$$

Example – 1 : **Find inverse of the following matrices** $\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$

Sol^n **: (i)** Given $A = \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}, |A| = 7$

$$A^{-1} = \frac{\text{adj } A}{|A|}, |A| \neq 0$$

So it has inverse

Adj (A)

Minor of 2, $M_{11} = 3$, Cofactor of 2, $C_{11} = 3$

Minor of –1, $M_{12} = 1$, Cofactor of –1, $C_{12} = -1$

Minoir of 1, $M_{21} = -1$, Cofactor of 1, $C_{21} = 1$

Minor of 3, $C_{22} = 2$, Cofactor of 3, $C_{22} = 2$

$$\text{adj}(A) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}}{7} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

Assignment

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

   Calculate (i) AB (ii) BA

2. Find the inverse of the following : $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

CHAPTER - 2

INTEGRAL CALCULUS

INTEGRATION AS INVERSE PROCESS OF DIFFERENTIATION

Integration is the process of inverse differentiation .The branch of calculus which studies about Integration and its applications is called Integral Calculus.

Let $F(x)$ and $f(x)$ be two real valued functions of x such that,

$$\frac{d}{dx}F(x) = f(x)$$

Then, $F(x)$ is said to be an anti-derivative (or integral) of $f(x)$.
Symbolically we write $\int f(x)\,dx = F(x)$.

The symbol $\int$ denotes the operation of integration and called the integral sign.
$'dx'$ denotes the fact that the Integration is to be performed with respect to x .The function $f(x)$ is called the Integrand.

INDEFINITE INTEGRAL

Let $F(x)$be an anti-derivative of $f(x)$.
Then, for any constant 'C',

$$\frac{d}{dx}\{F(x) + C\} = \frac{d}{dx}F(x) = f(x)$$

So,$F(x) + C$ is also an anti-derivative of $f(x)$, where C is any arbitrary constant. Then, $F(x) + C$ denotes the family of all anti-derivatives of $f(x)$, where C is an indefinite constant.

Therefore, $F(x) + C$ is called the Indefinite Integral of $f(x)$.
Symbolically we write

$$\int f(x)\,dx = F(x) + C,$$

Where the constant C is called the constant of integration. The function $f(x)$ is called the Integrand.

Example :-Evaluate $\int \cos x \; dx$.
Solution:-We know that

$$\frac{d}{dx}\sin x = \cos x$$

So, $$\int \cos x \; dx = \sin x + C$$

ALGEBRA OF INTEGRALS

1. $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$

2. $\int k\,f(x)dx = k\int f(x)dx,$ for any constant k.

3. $\int [a\,f(x) + b\,g(x)]\,dx = a\int f(x)dx + b\int g(x)dx,$
for any constant a & b

INTEGRATION OF STANDARD FUNCTIONS

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ , (n \neq -1)$
2. $\int \frac{1}{x} dx = \ln|x| + C$
3. $\int cosx\, dx = \sin x + C$
4. $\int \sin x\, dx = -\cos x + C$
5. $\int sec^2 x\, dx = \tan x + C$
6. $\int cosec^2 x\, dx = -\cot x + C$
7. $\int \sec x \tan x\, dx = \sec x + C$
8. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
9. $\int e^x dx = e^x + C$
10. $\int a^x dx = \frac{a^x}{\ln a} + C \ , (a > 0)$
11. $\int \tan x\, dx = \ln|\sec x| + C = -\ln|\cos x| + C$
12. $\int \cot x\, dx = \ln|\sin x| + C = -\ln|\operatorname{cosec} x| + C$
13. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
14. $\int \operatorname{cosec} x\, dx = \ln|\operatorname{cosec} x - \cot x| + C$
15. $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + C$
16. $\int \frac{1}{1+x^2} dx = tan^{-1}x + C$
17. $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1}x + C$
18. $\int \frac{1}{\sqrt{x^2+1}} dx = \ln\left|x + \sqrt{x^2+1}\right| + C$
19. $\int \frac{1}{\sqrt{x^2-1}} dx = \ln\left|x + \sqrt{x^2-1}\right| + C$

Example:- Evaluate $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

Solution:- $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

$= \int \frac{a^2 sin^2 x + b^2 cos^2 x}{4 sin^2 x . cos^2 x} dx$

$= \frac{a^2}{4} \int \frac{1}{cos^2 x} dx + \frac{b^2}{4} \int \frac{1}{sin^2 x} dx$

$= \frac{a^2}{4} \int sec^2 x\, dx + \frac{b^2}{4} \int cosec^2 x\, dx$

$= \frac{1}{4}[a^2 \tan x - b^2 \cot x] + C$

INTEGRATION BY SUBSTITUTION

When the integrand is not in a standard form, it can sometimes be transformed to integrable form by a suitable substitution.

The integral $\int f\{g(x)\}g^{'}(x)dx$ can be converted to
$\int f(t)dt$ by substituting $g(x)$by t.

So that, if $\int f(t)dt = F(t) + C$,then

$\int f\{g(x)\}g^{'}(x)dx = F\{g(x)\} + C.$

This is a direct consequence of CHAIN RULE.
For,

$$\frac{d}{dx}[F\{g(x)\} + C] = \frac{d}{dt}[F(t) + C].\frac{dt}{dx} = f(t).\frac{dt}{dx} = f\{g(x)\}g^{'}(x)$$

There is no fixed formula for substitution.

Example:- Evaluate $\int \cos(2 - 7x)\, dx$
Solution:- Put $t = 2 - 7x$
So that $\frac{dt}{dx} = -7 \Rightarrow dt = -7dx$

$$\therefore \quad \int \cos(2 - 7x)\, dx = \frac{-1}{7}\int \cos t\, dt$$
$$= \frac{-1}{7}\sin t + C$$
$$= \frac{-1}{7}\sin(2 - 7x) + C$$

INTEGRATION BY DECOMPOSITION OF INTEGRAND

If the integrand is of the form $\sin mx \cdot \cos nx$, $\cos mx \cdot \cos nx$ or $\sin mx \cdot \sin nx$, then we can decompose it as follows;

1. $\sin mx \cdot \cos nx = \frac{1}{2} \cdot 2 \sin mx \cdot \cos nx = \frac{1}{2}[\sin(m + n)x + \sin(m - n)x]$
2. $\cos mx \cdot \cos nx = \frac{1}{2}[\cos(m - n)x + \cos(m + n)x]$
3. $\sin mx \cdot \sin nx = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x]$

Similarly, in many cases the integrand can be decomposed into simpler form, which can be easily integrated.
Example:-Integrate $\int \sin 5x \cdot \cos 2x\, dx$

Solution:- $\int \sin 5x \cdot \cos 2x\, dx$
$$= \frac{1}{2}\int[\sin(5 + 2)x + \sin(5 - 2)x]\, dx$$
$$= \frac{1}{2}\int(\sin 7x + \sin 3x)\, dx$$
$$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x - \frac{1}{3}\cos 3x\right] + C$$
$$= -\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + C$$

Example:-Integrate $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x} dx$

Solution:- $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x} dx = \int \frac{2\sin 5x \cos x}{2\cos 5x \cos x} dx$
$$= \int \frac{\sin 5x}{\cos 5x} dx$$

Put t = cos 5x, so that $\frac{dt}{dx} = -5\, sin5x \Rightarrow dt = -5sin5x.dx$
$\therefore \int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x} dx = -\frac{1}{5}\int \frac{dt}{t} = -\frac{1}{5} ln|t| + C$

$$= -\frac{1}{5}\ln|\cos 5x| + C$$
$$= \frac{1}{5}\ln|\sec 5x| + C$$

INTEGRATION BY PARTS

This rule is used to integrate the product of two functions.
If u and v are two differentiable functions of x, then according to this rule have;

$$\int uv\, dx = u\int v\, dx - \int\left[\frac{du}{dx}\int v\, dx\right]dx$$

In words, Integral of the product of two functions

$$= \boldsymbol{first\ function} \times (\boldsymbol{Integral\ of\ second\ function}) - \boldsymbol{Integral\ of}(\boldsymbol{derivative\ of\ first} \times \boldsymbol{Integral\ of\ second})$$

The rule has been applied with a proper choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' functions .Usually from among exponential function(**E**),trigonometric function(**T**),algebraic function(**A**),Logarithmic function(**L**) and inverse trigonometric function(**I**),the choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' function is made in the order of **ILATE.**

Example:-Evaluate$\int x\sin x\, dx$

Solution:-$\int x\sin x\, dx$

$$= x\int \sin x\, dx - \int\left[\frac{dx}{dx}.\int \sin x\, dx\right]dx$$
$$= -x\cos x + \int \cos x\ dx$$
$$= \sin x - x\cos x + C$$

Example:-Evaluate$\int e^x \cos 2x\, dx$

Solution:- $\int e^x\cos 2x\, dx = e^x\cos 2x - \int e^x(-2\sin 2x)\, dx$

$$= e^x\cos 2x + 2\int e^x \sin 2x\, dx$$
$$= e^x\cos 2x + 2\left[e^x\sin 2x - 2\int e^x\cos 2x\, dx\right]$$
$$= e^x\cos 2x + 2\, e^x\sin 2x - 4\int e^x \cos 2x\, dx + K$$

So,$5\int e^x\cos 2x = e^x[\cos 2x + 2\sin 2x] + K$

$\therefore \int e^x \cos 2x\, dx = \frac{e^x}{5}[\cos 2x + 2\sin 2x] + C$ (where $= {K}/{2}$)

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

The irrational forms $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$ can be simplified to radical free functions as integrand by putting $x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$ respectively.
The substitution $x = a\tan\theta$ can be used in case of presence of $x^2 + a^2$ in the integrand, particularly when it is present in the denominator.

ESTABLISHMENT OF STANDARD FORMULAE

1. $\int\frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right| + C$

5. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

Solutions:

1. Let $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$ and $\theta = sin^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2 sin^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \int d\theta = \theta + C = sin^{-1}\frac{x}{a} + C$

2. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\ d\theta$ and $\theta = tan^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta\ d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\sec^2\theta\ d\theta}{a^2(\tan^2\theta + 1)} = \int \frac{asec^2\theta}{a^2sec^2\theta} d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$

$= \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\ d\theta$ and $\theta = sec^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\ d\theta}{a\sec\theta\sqrt{a^2sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\, a\tan\theta} d\theta = \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + C = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\ d\theta$.

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\ d\theta}{\sqrt{a^2tan^2\theta + a^2}} = \int \frac{asec^2\theta}{a\sec\theta} d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + K$

$= \ln\left|\sqrt{tan^2\theta + 1} + \tan\theta\right| + K = \ln\left|\sqrt{\frac{x^2}{a^2} + 1} + \frac{x}{a}\right| + K$

$= \ln\left|\frac{x + \sqrt{x^2+a^2}}{a}\right| + K$

$= \ln\left|x + \sqrt{x^2+a^2}\right| + K - \ln|a|$

$= \ln\left|x + \sqrt{x^2+a^2}\right| + C$ (Where $C = K - \ln|a|$)

5. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\ d\theta$

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\ d\theta}{\sqrt{a^2sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta} d\theta = \int \sec\theta\ d\theta$

$= \ln|\sec\theta + \tan\theta| + K = \ln\left|\sec\theta + \sqrt{sec^2\theta - 1}\right| + K$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + K$

$= \ln\left|\frac{x + \sqrt{x^2-a^2}}{a}\right| + K$

$= \ln\left|x + \sqrt{x^2-a^2}\right| + K - \ln|a|$

$= \ln\left|x + \sqrt{x^2-a^2}\right| + C$ (Where $C = K - \ln|a|$)

SOME SPECIAL FORMULAE

1. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$

2. $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2+a^2}\right| + C$

3. $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right| + C$

Solutions:

1. $\int \sqrt{a^2-x^2}dx = \int 1 \cdot \sqrt{a^2-x^2}dx$

$= x\sqrt{a^2-x^2} - \int x\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)dx$

$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \sqrt{a^2-x^2}dx$

$\therefore\ 2\int \sqrt{a^2-x^2}dx = x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$

$= x\sqrt{a^2-x^2} + a^2 sin^{-1}\frac{x}{a} + K$

$\therefore\ \int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$ (Where $C = \frac{K}{2}$)

2. $\int \sqrt{x^2+a^2}dx = \int 1 \cdot \sqrt{x^2+a^2}dx$

$= x\sqrt{x^2+a^2} - \int x\left(\frac{2x}{2\sqrt{x^2+a^2}}\right)dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$

$= x\sqrt{x^2+a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx$

$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}dx + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$

$\therefore\ 2\int \sqrt{x^2+a^2}dx = x\sqrt{x^2+a^2} + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$

So, $2\int \sqrt{x^2+a^2}dx = x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + K$

$\therefore\ \int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

(Where $C = \frac{K}{2}$)

3. $\int \sqrt{x^2-a^2}dx = \int 1 \cdot \sqrt{x^2-a^2}dx$

$= x\sqrt{x^2-a^2} - \int x\left(\frac{2x}{2\sqrt{x^2-a^2}}\right)dx$

$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$

$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}dx$

$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}dx + a^2\int \frac{dx}{\sqrt{x^2-a^2}}$

$\therefore\ 2\int \sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$

So, $2\int \sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2 \ln\left|x+\sqrt{x^2-a^2}\right| + K$

$\therefore\ \int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

(Where $C = \frac{K}{2}$)

METHOD OF INTEGRATION BY PARTIAL FRACTIONS

If the integrand is a proper fraction $\frac{P(x)}{Q(x)}$, then it can be decomposed into simpler partial fractions and each partial fraction can be integrated separately by the methods outlined earlier.

SOME SPECIAL FORMULAE

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

Solutions:

1. We have, $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$

$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + C$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

2. We have, $\frac{1}{a^2-x^2} = \frac{1}{(a+x)(a-x)}$

$= \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)$

$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int\left(\frac{1}{a+x} + \frac{1}{a-x}\right)dx$

$= \frac{1}{2a}[\ln|a+x| - \ln|a-x|] + C$

$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

Example:- Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

Solution:- Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$ ---------------(1)

Multiplying both sides of (1) by $(x-1)^2(x+3)$,

$\Rightarrow\ x^2+1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$ ---------------(2)

Putting $x = 1$ in (2),we get,$B = \frac{1}{2}$

Putting $x = -3$ in (2),we get,$10 = 16C \Rightarrow C = \frac{5}{8}$

Equating the co-efficients of x^2 on both sides of (2), we get

$1 = A + C \Rightarrow A = 1 - \frac{5}{8} = \frac{3}{8}$

Substituting the values of A,B &C in (1) ,we get

$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}\cdot\frac{1}{x+3}$

$\therefore \int \frac{x^2+1}{(x-1)^2(x+3)}dx$

$= \frac{3}{8}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{(x-1)^2} + \frac{5}{8}\int\frac{dx}{x+3}$

$= \frac{3}{8}\ln|x-1| + \frac{5}{8}\ln|x+3| - \frac{1}{2(x-1)} + C$

Example:- Evaluate $\int \frac{x}{(x-1)(x^2+4)}dx$

Solution:- Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ ---------------(1)

Multiplying both sides of (1) by $(x-1)(x^2+4)$,we get

$x = A(x^2+4) + (Bx+C)(x-1)$--------------(2)

Putting $x = 1$ in (2), we get,$A = \frac{1}{5}$

Putting $x = 0$ in (2), we get,$0 = 4A - C \Longrightarrow C = 4A \Longrightarrow C = \frac{4}{5}$

Equating the co-efficients of x^2 on both sides of (2), we get

$0 = A + B \Rightarrow B = -\frac{1}{5}$

Substituting the values of A, B and C in (1) we get

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

$\therefore \quad \int \frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{x-4}{x^2+4}dx$

$= \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$

$= \frac{1}{5}\int \frac{dx}{x-1} + \frac{1}{10}\int \frac{2xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$

$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| + \frac{2}{5}tan^{-1}\left(\frac{x}{2}\right) + C$

Example:- Evaluate $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$

Solution:- Let $x^2 = y$ $Then \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ ---------------(1)

Multiplying both sides of (1) by $(y+1)(y+4)$,we get

$y = A(y+4) + B(y+1)$-----------(2)

Putting $y = -1$ and $y = -4$ successively in (2),we get,$A = -\frac{1}{3}$ and$B = \frac{4}{3}$

Substituting the values of A and B in (1),we get

$\frac{\square}{(\square+1)(\square+4)} = -\frac{1}{3(\square+1)} + \frac{4}{3(\square+4)}$

Replacing □ by $\square^2$, we obtain

$\frac{\square^2}{(\square^2+1)(\square^2+4)} = -\frac{1}{3(\square^2+1)} + \frac{4}{3(\square^2+4)}$

$\therefore \quad \int \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{-1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$

$= -\frac{1}{3}tan^{-1}x + \frac{2}{3}tan^{-1}\left(\frac{x}{2}\right) + C$

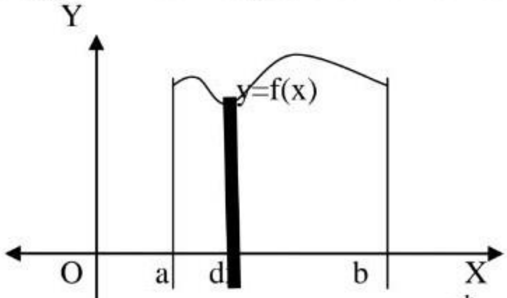
DEFINITE INTEGRAL

If $f(x)$ is a continuous function defined in the interval [a,b] and$F(x)$ is an anti-derivative of $f(x)$ $i.e.,\frac{dF(x)}{dx} = f(x)$,then the define integral of $f(x)$ over [a,b] is denoted by

$\int_a^b f(x)dx$ and is equal to $F(b) - F(a)$

$i.e., \int_a^b f(x)dx = F(b) - F(a)$

The constants a and b are called the limits of integration. 'a' is called the lower limit and 'b' the upper limit of integration.The interval [a,b] is called the interval of integration.



Geometrically, the definite integral $\int_a^b f(x)dx$ is the AREA of the region bounded by the curve $y = f(x)$ and the lines $x = a, x = b$ and x-axis.

EVALUATION OF DEFINITE INTEGRALS

To evaluate the definite integral $\int_a^b f(x)dx$ of a continuous function $f(x)$ defined on [a, b], we use the following steps.

Step-1:-Find the indefinite integral $\int f(x)dx$

Let $\int f(x)dx = F(x)$

Step-2:-Then, we have

$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$$

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$

i.e., definite integral is independent of the symbol of variable of integration.

3. $\int_a^b f(x)dx = \int_a^c f(x)dx = \int_c^b f(x)dx, a < c < b$

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx, a > 0$

5. $\int_{-a}^a f(x)dx = \begin{bmatrix} 2\int_0^a f(x)dx, & if f(-x)=f(x) \\ 0, & if f(-x)=-f(x) \end{bmatrix}$

6. $\int_0^{2a} f(x)dx = \begin{bmatrix} 2\int_0^a f(x)dx, & if f(2a-x)=f(x) \\ 0, & if f(2a-x)=-f(x) \end{bmatrix}$

Example:- Evaluate $\int_0^1 x tan^{-1}x \, dx$

Solution:-We have, $\int x\tan^{-1}x\, dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{x^2+1}$

$=\frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x$

$=\frac{(x^2+1)}{2}\tan^{-1}x - \frac{x}{2}$

$\therefore \quad \int_0^1 x\tan^{-1}x\, dx = \left[\frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2}\right]_0^1$

$=\tan^{-1}1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$

Example:- Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx$

Solution:-Let $I=\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx$

$=\int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\, dx$

$=\int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\, dx$

$\therefore\ 2I=I+I=\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\, dx = \int_0^{\pi/2} \frac{(\sin x+ \cos x)}{(\sin x+\cos x)}\, dx$

$=\int_0^{\pi/2} dx = x\Big]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore\ I=\frac{\pi}{4}$

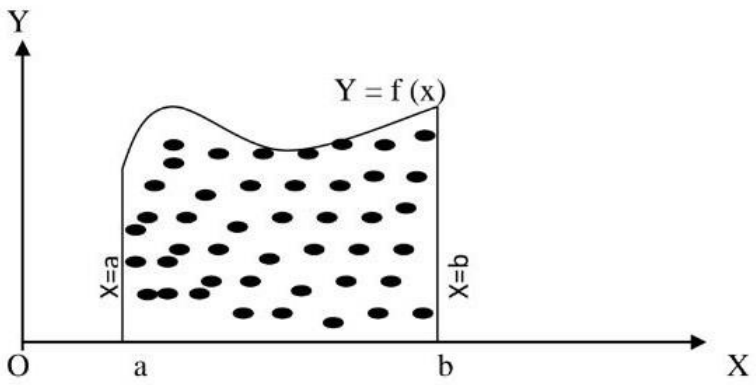
$\therefore \quad \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx = \frac{\pi}{4}$

AREA UNDER PLANE CURVES

DEFINITION-1:-

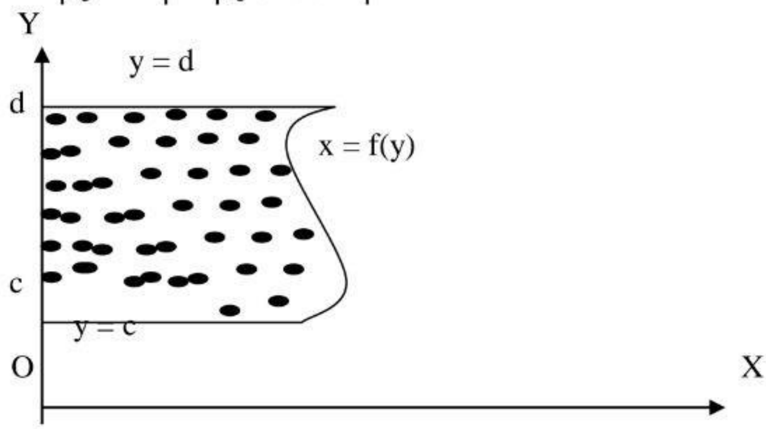
Area of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a, x = b$ is defined by

Area$=\left|\int_a^b y dx\right| = \left|\int_a^b f(x)dx\right|$



DEFINITION-2:-Area of the region bounded by the curve $x = f(y)$, the Y-axis and the lines $y = c, y = d$ is defined by

Area=$\left|\int_c^d x dy\right| = \left|\int_c^d f(y) dy\right|$



Example:-Find the area of the region bounded by the curve $y = e^{3x}$, x-axis and the lines $x = 4, x = 2$.

Solution:-The required area is defined by

$$A=\int_2^4 e^{3x} dx = \frac{1}{3}e^{3x}\Big]_2^4 = \frac{1}{3}(e^{12x} - e^{6x})$$

Example:-Find the area of the region bounded by the curve $xy = a^2$, y-axis and the lines $y = 2, y = 3$

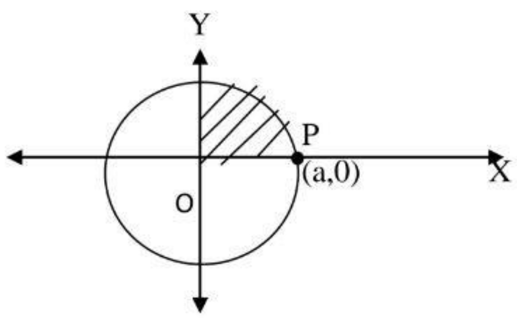
Solution:- $We\ have,\ xy = a^2 \Rightarrow x = \frac{a^2}{y}$

∴ The required area is defined by

$$A=\int_2^3 x dy = a^2 \int_2^3 \frac{dy}{y} = [a^2 \ln y]_2^3 = a^2(\ln 3 - \ln 2) = a^2 \ln\left(\frac{3}{2}\right)$$

Example:-Find the area of the circle $x^2+y^2 = a^2$

Solution:-We observe that, $y = \sqrt{a^2-x^2}$ in the first quadrant.



∴ The area of the circle in the first quadrant is defined by,

$A_1 = \int_0^a \sqrt{a^2-x^2}\, dx$

CHAPTER - 3

CO-ORDINATE GEOMETRY

STRAIGHT LINE

CO-ORDINATE SYSTEM

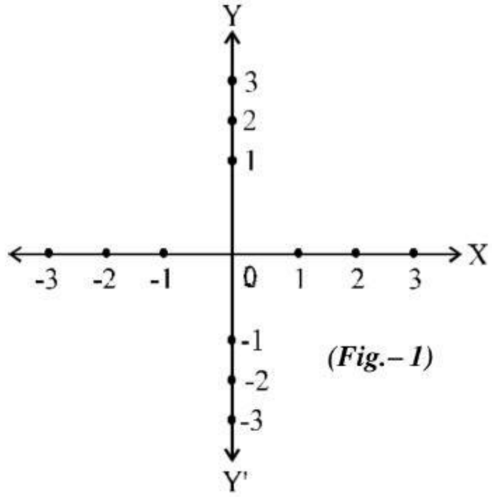
We represent each point in a plane by means of an ordered pair of real numbers, called co-ordinates. The branch of mathematics in which geometrical problems are solved through algebra by using the co-ordinate system, is known as co-ordinate geometry or analytical geometry.

Rectangular co-ordinate Axes

Let X'OX and YOY' be two mutually perpendicular lines (called co-ordinate axes), intersecting at the point O. ***(Fig.1)***.We call the point O, the origin, the horizontal line X'OX, the x-axis and the vertical line YOY', the y-axis.

We fix up a convenient unit of length and starting from the origin as zero, mark. distances on x-axis as well as y-axis. X,

The distance measured along OX and OY are taken as positive while those along OX' and OY' are considered negative.



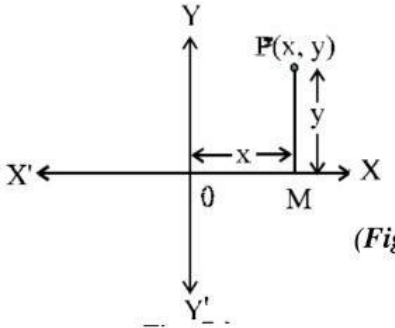
(Fig.– 1)

Cartesian co-ordinates of a point

Let X'OX and YOY' be the co-ordinate axes and let P be a point in the Euclidean plane ***(Fig.2)***. From P draw PM ⊥ X'OX.

Let OM = x and PM = y, Then the ordered pair (x, y) represents the cartesian co-ordinates of P and we denote the point by P(x, y). The number x is called the x-co-ordinate or abscissa of the point P, while y is known as its y-co-ordinate or ordinate.

Thus, for a given point the abscissa and the ordinate are the distances of the given point from y- axis and x-axis respectively.



(Fig.– 2)

Quadrants

The co-ordinate axes X'OX and Y'OY divide the plane in to four regions, called quadrants.

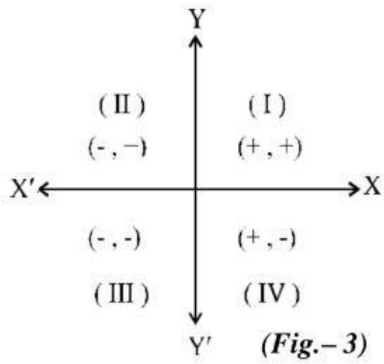
The regions XOY, YOX', X'OY' and Y'OX are known as the first, the second, the third and the fourth quadrant respectively. ***(Fig.3)*** In accordance with the convention of signs defined above for a point (x, y) in different quadrants we have

1st quadrant : $x > 0$ and $y > 0$

2nd quadrant : $x < 0$ and $y > 0$

3rd quadrant : $x < 0$ and $y < 0$

4th quadrant : $x > 0$ and $y < 0$

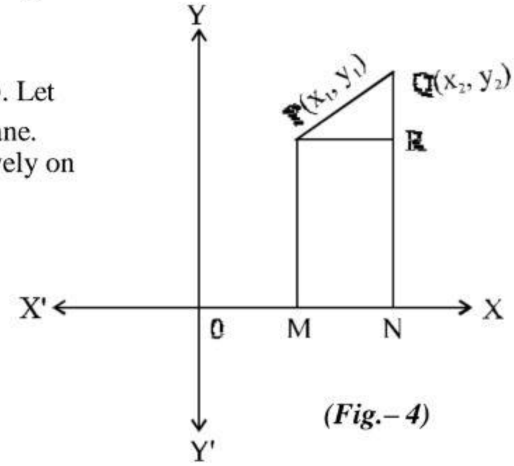


(Fig.– 3)

DISTANCE BETWEEN TWO GIVEN POINTS

The distance between any two points in the plane is the length of the line segment joining them.

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



(Fig.– 4)

Proof : Let X'OX and YOY' be the co-ordinate axes *(Fig.4)*. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points in the plane. From P and Q draw perpendicular PM and QN respectively on the x-axis. Also draw PR ⊥ QN.

Then, $OM = x_1$, $ON = x_2$

$PM = y_1$ & $QN = y_2$

$\therefore PR = MN = ON - OM = x_2 - x_1$ and

$QR = QN - RN = QN - PM = y_2 - y_1$

Now from right angled triangle PQR,

we have $PQ^2 = PR^2 + QR^2$ [by Pythagoras theorem]

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

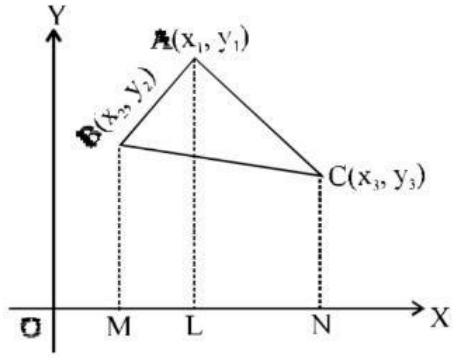
$\therefore |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Cor : The distance of a point P(x, y) from the origin O (0, 0) is

$= \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$

Area of a triangle :

Let ABC be a given triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. From the vertices A, B and C draw perpendiculars AL, BM and CN respectively on x-axis. *(Fig.5)*.



(Fig.– 5)

Then, $ML = x_1 - x_2$; $LN = x_3 - x_1$ and $MN = x_3 - x_2$

$\therefore$ Area of Δ ABC

= area of trapezium ALMB + area of trapezium ALNC – area of trapezium BMNC

$= \frac{1}{2}(AL+BM).ML + \frac{1}{2}(AL + CN).LN - \frac{1}{2}(MB+CN).MN$

$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$

$= \frac{1}{2}[x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2 + x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 - x_3y_2 - x_3y_3 + x_2y_2 + x_2y_3]$

$= \frac{1}{2}[x_1y_2 - x_2y_1 + x_3y_1 - x_1y_3 - x_3y_2 + x_2y_3]$

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

In determinant form, we may write

$$\text{Area of } \Delta\, ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition for collinearity of Three points :

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are colliner, i.e. lie on the same straight line, if the area of Δ ABC is zero. So the required condition for A, B, C to be collinear is that

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$$

$$\Rightarrow \quad x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

Formula for Internal Divisions :

The co-ordinates of a point P which divides the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio m : n are given by

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}$$

Example – 1 : **In what ratio does the point (3, –2) divide the line segment joining the points (1, 4) and (– 3, 16) :**

Solution : Let the point C (3, –2) divide the segment joining A(1, 4) and B (–3, 16) in the ratio K: 1

The co-ordinates of 'C' are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1}\right)$

But we are given that the point C is (3, –2)

$\therefore$ We have $\frac{-3k+1}{k+1} = 3$

or $-3k + 1 = 3k + 3$

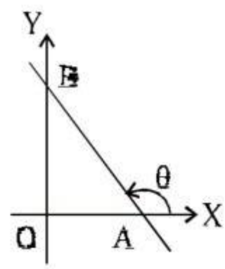
or $-6k = 2$

$\therefore$ $k = -\frac{1}{3}$

$\therefore$ C divides AB in the ratio 1 : 3 externally.

SLOPE OF A LINE

Angle of Inclination : The angle of inclination or simply the inclination of a line is the angle θ made by the line with positive direction of x-axis, measured from it in anticlock wise direction ***(Fig. 6)***.



(Fig.– 6)

Slope or gradient of a line : If θ is the inclination of a line, then the value of $\tan\theta$ is called the slope of the line and is denoted by m.

CONDITIONS OF PARALLELISM AND PERPENDICULARITY

1. Two lines are parallel if and only if their slopes are equal.

2. Two lines with slope m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$

3. The slope of a line passing through two given points (x_1, y_1) and (x_2, y_2) is given by $m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

4. The equation of a line with slope m and making an intercept 'c' on y-axis is given by y = mx + c.

Proof : Let AB be the given line with inclination θ so that tan θ = m. Let it intersect the y-axis at C so that OC = c. ***(Fig.7)***

Let it intersect the x-axis at A.

Let P(x, y) be any point on the line.

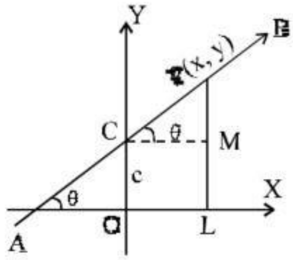
Draw PL perpendicular to x-axis and CM ^ PL

Clearly, Ð MCP = Ð OAC = q

CM = OL = x ;

and PM = PL – ML = PL – OC = y – c

Now, from rt. angled D PMC



(Fig.– 7)

We get $\tan\theta = \frac{PM}{CM}$ or $m = \frac{y - c}{x}$

or $y = mx + c$, which is required equation of the line.

5. The equation of a line with slope m and passing through a point (x_1, y_1) is given by $(y - y_1) = m(x - x_1)$

6. The equation of a line through two given points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

7. The equation of a straight line which makes intercepts of length 'a' and 'b' on x-axis and y-axis respectively, is $\frac{x}{a} + \frac{y}{b} = 1$

Proof : Let AB be a given line meeting the x-axis and y-axis at A and B respectively ***(Fig.8)***. Let OA = a and OB = b

Then the co-ordinates of A, B are A(a, 0) and B(0, b)

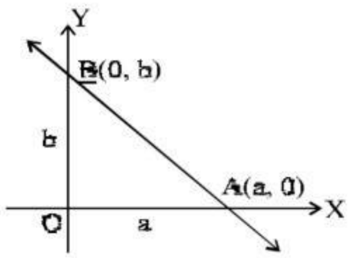
∴ The equation of the line joining A & B is

$$(y - 0) = \frac{b - 0}{0 - a}(x - a)$$

Þ $y = \frac{-b}{a}(x - a)$

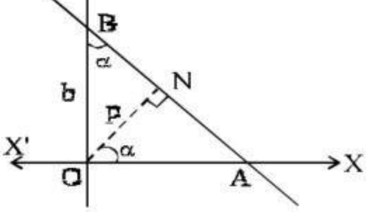
Þ $\frac{y}{b} = \frac{-x}{a} + 1$

Þ $\frac{x}{a} + \frac{y}{b} = 1$



(Fig.– 8)

8. Let P be the length of perpendicular from the origin to a given line and a be the angle made by this perpendicular with the positive direction of x-axis. Then the equation of the line is given by x cos a + y sin a = P



(Fig.– 9)

Conditions for two lines to be coincident, parallel, perpendicular or Intersect :

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) conicident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$;

(ii) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Perpendicular, if $a_1a_2 + b_1b_2 = 0$;

(iv) Intersecting, if they are neither coincident nor parallel.

Example – 1 : **Find the equation of the line which passes through the point (3, 4) and the sum of its intercept on the axes is 14.**

Sol^n : Let the intercept made by the line on x-axis be 'a' and 'y'- axis be 'b'

i.e. $a + b = 14$ i.e, $b = 14 - a$

∴ Equation of the line is given by

$$\frac{x}{a} + \frac{y}{14 - a} = 1 \text{ .... (i)}$$

As the point (3, 4) lies on it, we have

$$\frac{3}{a} + \frac{4}{14 - a} = 1$$

or $3(14 - a) + 4a = 14a - a^2$

or $42 - 3a + 4a = 14a - a^2$

or $a^2 - 13a + 42 = 0$

or $(a - 7)(a - 6) = 0$

or $a = 7$ or $a = 6$

Putting these values of a in (i)

$\frac{x}{7} + \frac{y}{7} = 1$ or $x + y = 7$

and $\frac{x}{6} + \frac{y}{8} = 1$ or $4x + 3y = 24$

Example – 2 : **Find the equation of the line passing through (–4, 2) and parallel to the line $4x - 3y = 0$**

Sol^n : Any line passing thorugh (–4, 2) whose equation is given by

$(y - 2) = m(x + 4)$ .... (i)

and parallel to the given line $4x - 3y = 0$

whose slope is $y = \frac{4}{3}x$

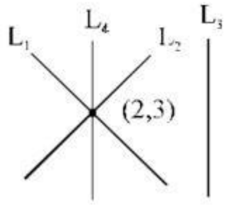
Here 'm' = $\frac{4}{3}$

It's equation is

$(y - 2) = \frac{4}{3}(x + 4)$

$3y - 6 = 4x + 16$

or $4x - 3y + 22 = 0$



(Fig.– 10)

Example – 3 : **Find the equation of the line passing through the intersection of 2x – y – 1 = 0 and 3x – 4y + 6 = 0 and parallel to the line x + y – 2 = 0**

Sol*ⁿ* *:* Point of intersection of 2x – y – 1 = 0 and 3x – 4y + 6 = 0

$$\left(\frac{-1\times 6-(-4)(-1)}{2(-4)-3(-1)}, \frac{(-1)\times 3-6(2)}{2(-4)-3(-1)}\right)$$

$$=\left(\frac{-6-4}{-8+3}, \frac{-3-12}{-8+3}\right)=\left(\frac{-10}{-5}, \frac{-15}{-5}\right)=(2, 3)$$

Any line parallel to the line x + y – 2 is given by x + y + k = 0.... (i)

Since the line passes through (2, 3) hence it satisfies the equation (i)

So, 2 + 3 + k = 0

$\Rightarrow$ k = –5

Now putting the value of k in equation (i), we get x + y – 5 = 0

∴ Equation of the line is x + y – 5 = 0

Assignment

1. Find the equation of a line parallel to 2x + 4y – 9 = 0 and passing through the point (–2, 4)
2. Find the co-ordinates of the foot of the perpendicular from the point (2, 3) on the line 3x – 4y + 7 = 0
3. Find the equation of the line through the point of intersection of 3x + 4y – 7 = 0 and x – y + 2 = 0 and which is parallel to the line 5x – y + 11 = 0

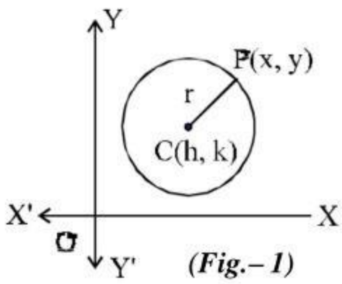
r] r

CIRCLE

A circle is the locus of a point which moves in a plane in such a way that it's distance from a fixed point is always constant.

The fixed point is called the centre of the circle and the constant distance is called its radius.

Equation of a circle (Standard form)



(Fig.– 1)

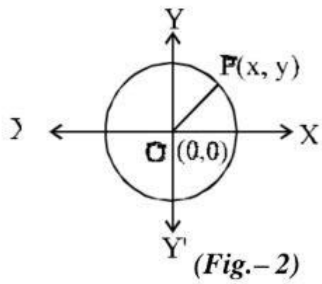
Let C(h, k) be the centre of a circle with radius 'r' and let P (x, y) be any point on the circle *(Fig.1)*.

Then $CP = r \Rightarrow CP^2 = r^2$

$\Rightarrow (x-h)^2 + (y-k)^2 = r^2$

Which is required equation of the circle.

Cor. The equation of a circle with the centre at the origin and radius r, is $x^2 + y^2 = r^2$ ***(Fig.2)***.



(Fig.– 2)

Proof : Let O (0, 0) be the centre and r be the radius of a circle and let P (x, y) be any point on the circle.

Then $OP = r \Rightarrow OP^2 = r^2$

$\Rightarrow (x-0)^2 + (y-0)^2 = r^2$

$\Rightarrow x^2 + y^2 = r^2$

Example – 1*. Find the equation of a circle with centre (–3, 2) and radius 7.** ***Sol^n **:** The required equation of the circle is

$[x-(-3)]^2 + (y-2)^2 = 7^2$

or $(x+3)^2 + (y-2)^2 = 49$

or $x^2 + y^2 + 6x - 4y - 36 = 0$

***Example – 2.* Find the equation of a circle whose centre is (2, –1) and which passes through (3, 6)**

Sol^n **:** Since the point P (3, 6) lies on the circle, its distance from the centre C (2, –1) is therefore equal to the radius of the circle.

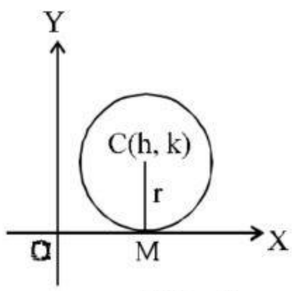
$\therefore$ Radius $= CP = \sqrt{(3-2)^2 + (6+1)^2} = \sqrt{50}$

So, the required equation of the circle is

$(x-2)^2 + (y+1)^2 = 50$

or $x^2 + y^2 - 4x + 2y - 45 = 0$

***Example – 3*. Find the equation of a circle with centre (h, k) and touching the x-axis *(Fig.3)*.**



(Fig.– 3)

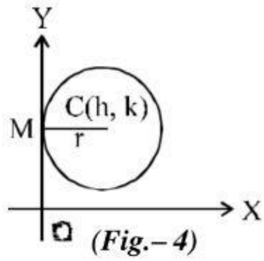
Sol^n **:** Clearly, the radius of the circle = CM = r = k

So, the required equation

$(x-h)^2 + (y-k)^2 = k^2$

or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$

***Example – 4*. Find the equation of a circle with centre (h,k) and touching y-axis*(Fig.4)*.**



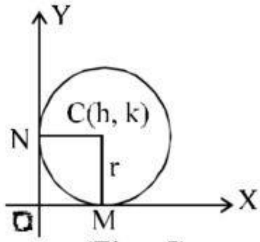
(Fig.– 4)

***Sol**^{n}* **:** Clearly, the radius of the circle = CM = r = h

So, the required equation is $(x-h)^2 + (y-k)^2 = h^2$

or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$

***Example – 5*. Find the equation of a circle with centre (h,k) and touching both the axes *(Fig.5)*.**



(Fig.– 5)

***Sol**^{n}* **:** Clearly, radius, CM = CN = r

i.e. h = k = r (say)

$\therefore$ the equation of the circle is $(x-r)^2 + (y-r)^2 = r^2$ or $x^2 + y^2 - 2r(x+y) + r^2 = 0$

GENERAL EQUATION OF A CIRCLE

Theorem : The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ And, every such equation represents a circle.

Proof : The standard equation of a circle with centre (h, k) and radius r is given by

$(x-h)^2 + (y-k)^2 = r^2$

Or $x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

This is of the form

$x^2 + y^2 + 2gx + 2fy + c = 0$

Where $h = -g, k = -f$ and $c = (h^2 + k^2 - r^2)$

Conversely, let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the given condition.

Then, $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = (g^2 + f^2 - c)$

$\Rightarrow (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

$\Rightarrow [x-(-g)]^2 + [y-(-f)]^2 = \left[\sqrt{g^2 + f^2 - c}\right]^2$

$\Rightarrow (x-h)^2 + (y-k)^2 = r^2$

Where $h = -g$, $k = -f$ and $r = \sqrt{g^2 + f^2 - c}$

This shows that the given equation represents a circle with centre (–g, –f) and radius.

$= \sqrt{g^2 + f^2 - c}$, provided $g^2 + f^2 > c$.

EQUATION OF A CIRCLE WITH GIVEN END POINTS OF A DIAMETER

Theorem : The equation of a circle described on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_1)$ as a diameter, is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

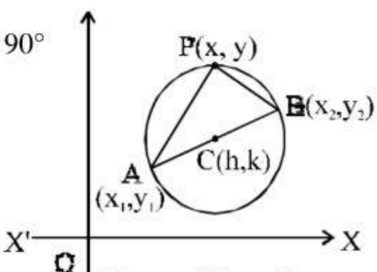
Proof : Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end point of a diameter of the given circle and let P (x, y) be any point on the circle ***(Fig.6)***. .Y

Since the angle in a semi-circle is a right angle, we have $\angle APB = 90°$

Now slope of AP = $\left(\frac{y - y_1}{x - x_1}\right)$

And, slope of BP = $\left(\frac{y - y_2}{x - x_2}\right)$

Since $AP \perp BP$, we have



(Fig.– 6)

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_1}\right) = -1$$

Or $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

***Example – 1*. Find the equation of a circle whose end points of diameter are (3, 4) and 3, –4)**

Soln. : The required equation of the circle is $(x - 3)(x + 3) + (y - 4)(y + 4) = 0$

i.e. $x^2 - 9 + y^2 - 16 = 0$

or $x^2 + y^2 = 25$

***Example – 2*. Find the centre and radius of the circle.**

$$x^2 + y^2 - 6x + 4y - 36 = 0$$

***Soln*. :** Comparing the equation with

$x^2 + y^2 + 2gx + 2fy + c = 0$

We get $2g = -6$, $2f = 4$ and $c = -36$

or $g = -3$, $f = 2$ and $c = -36$

\ Centre of the circle is $(-g, -f)$, i.e. (3, –2) And radius of the circle.

$= \sqrt{g^2 + f^2 - c} = \sqrt{9 + 4 + 36} = 7$

Assignment

1. Find the centre and radius of each of the following circles
   $x^2 + y^2 + x - y - 4 = 0$
2. Find the equation of the circle whose centre is (–2, 3) and passing through origin
4. Find the equation of the circle having centre at (1, 4) and passing through (–2, 1).
4. Find the equation of the circle passing through the points (1, 3) (2, –1) and (–1, 1).

r] r

CHAPTER - 4

VECTOR CALCULUS

Introduction :

At present vector methods are used in almost all branches of science such as Mechanics, Mathematics, Engineering, physics and so on. Both the theory and complicated problems in these subjects can be discussed in a simple manner with the help of vectors. It is a very useful tool in the hands of scientists.

Physical quantities are divided into two category scalar quantities and vector quantities. Those quantities which have any magnitude and which and not related to any fixed direction scalars. Example of scalars are mass volume density, work, temperature etc. Second kind of quantities are those which have both magnitude and direction. Such quantities are vectors. Displacement, velocity, acceleration, momentum weight, force etc. are examples of vector quantities.

Representation of vectors :

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end indicates the direction of vector. A vector denoted by $\overrightarrow{PQ}$, is determined by two points P, Q such that the direction of the vector is the length of the straight line PQ and its direction is that from P to Q. The point P is called initial point of vector $\overrightarrow{PQ}$ and Q is called terminal points.

Note : The length (magnitude or modulus) of $\overrightarrow{AB}$ or $\vec{a}$ generally denoted my $|\overrightarrow{AB}|$ or $|\vec{a}|$ thus $|\vec{a}|$ = length (magnitude or modulus ? or vector $\vec{a}$)

Types of vectors :

(i) Zero vector or null vector : A vector whose initial so terminal points are coincident is called zero or the null vector. The modulus of a null vector is zero.

(ii) Unit vector : A vector whose modulus in unity, is called a unit vector. The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$. Thus $|\hat{a}| = 1$

(iii) Like and unlike vector : Vectors are said to be like when they have same sense of direction and unlike when they have opposite directions.

(iv) Collinear or Parallel vector : Vectors having the same or parallel supports are called collinear vectors.

(v) Co-initial vectors: Vectors having the same initial point are called co-initial vector.

(vi) Co-planner vector : A system of vector and said to be co-planner in their supports are parallel to the same plane.

(vii) Negative of a vector : The vector which has the same Magnitude as the vector $\vec{a}$ but opposite direction, is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. There if $\overrightarrow{PQ} = \vec{a}$ then $\overrightarrow{QP} = -\vec{a}$.

Operations on Vectors

Addition of Vectors :

Tringle Law of Addition of Two Vectors :

The law states that if two vectors are represented by the two sides of a triangle, taken in order, then their sum (or resultant) is represented by the third side of the triangle but in the reverse order.

Let $\vec{a}$, $\vec{b}$ be the given vectors. Let the vector $\vec{a}$ be represented by the directed segment $\overrightarrow{OA}$ and the vector $\vec{b}$ be the directed segment $\overrightarrow{AB}$ so that the terminal point A of $\vec{a}$ is the initial

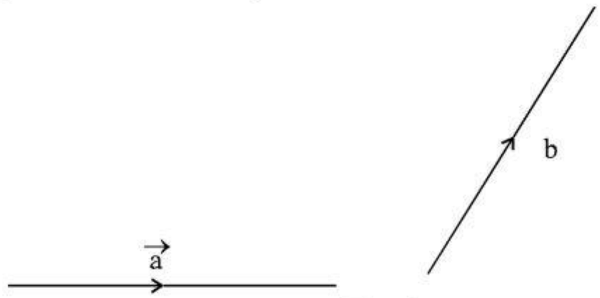


Fig. 1

point of $\vec{b}$. Then the directed segment OB (i.e. $\overrightarrow{OB}$) represents the sum (or resultant) or $\vec{a}$ and $\vec{b}$ and is written as $\vec{a} + \vec{b}$ ***(fig. 2)***

Thus, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

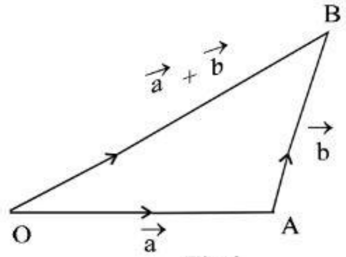


Fig.2

Note : 1. The method of drawing a triangle in order to define the vector sum ($\vec{a} + \vec{b}$) is called **triangle law of addition of the vectors.**

2. Since any side of a triangle is less than the sum of the other two sides.

$\therefore$ Modulus of $\overrightarrow{OB}$ is not equal to the sum of the modulus of $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

Parallelogram Law of Vectors

If two vectors $\vec{a}$ and $\vec{b}$ are represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common initial point.

Let $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Then

$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ *.(fig.3)*

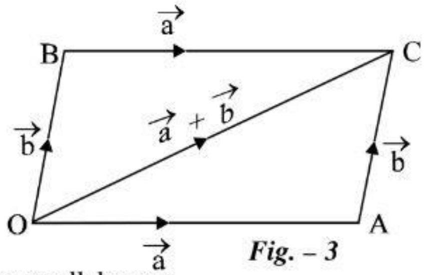


Fig. – 3

i.e. Their sum $\vec{a} + \vec{b}$ is respesented by the diagonal $\overrightarrow{OC}$ of the parallelogram.

Polygon Law of addition of Vectors

To add n vectors $\vec{a}_1, \vec{a}_2, \ldots\ldots \vec{a}_n$ we choose O as an origin *(fig.4)* and draw.

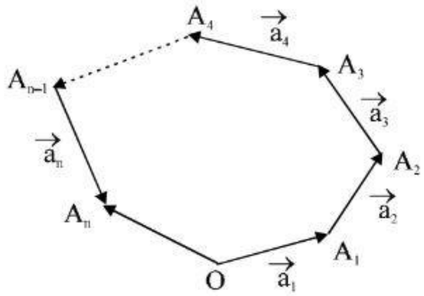


Fig. – 4

$\overrightarrow{OA_1} = \vec{a}_1, \overrightarrow{A_1A_2} = \vec{a}_2, \ldots\ldots \overrightarrow{A_{n-1}A_n} = \vec{a}_n$

$\therefore\ \vec{a}_1 + \vec{a}_2 + \ldots\ldots + \vec{a}_n$

$= \overrightarrow{OA_1} + \overrightarrow{A_1A_2} + \ldots\ldots + \overrightarrow{A_{n-1}A_n}$

$= (\overrightarrow{OA_1} + \overrightarrow{A_1A_2}) + \overrightarrow{A_2A_3} + \ldots\ldots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_2} + \overrightarrow{A_2A_3} + \ldots\ldots + \overrightarrow{A_{n-1}A_n}$

$= \overrightarrow{OA_3} + \overrightarrow{A_3A_4} + \ldots\ldots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n}$

Hence the sum of vectors $\vec{a}_1, \vec{a}_2 \ldots\ldots \vec{a}_n$ is represented by $\overrightarrow{OA_n}$. This method of vector addition is called "polygon law of addition of vectors.

Corollary : From the polygon law of addition of vectors, we have

$\overrightarrow{OA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots\ldots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n} = -\overrightarrow{A_nO}$

$\overrightarrow{OA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots\ldots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nO} = \overrightarrow{A_nO}$ (Null vector)

$\therefore$ The sum of vectors determined by the sides of any polygon taken in order is zero.

Properties of Vectors Addition

(1) Vector Addition is Commulative :

If $\vec{a}$ and $\vec{b}$ be any two vectors, then

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Proof : Let the vectors $\vec{a}$ and $\vec{b}$ be represented by the directed segments $\overrightarrow{OA}$ and $\overrightarrow{AB}$ respectively so that *(fig.4)*

$\vec{a} = \overrightarrow{OA},\ \vec{b} = \overrightarrow{AB}$

Now $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} \Rightarrow \overrightarrow{OB} = \vec{a} + \vec{b}$

Complete the || gm OABC

Then $\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$ and $\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$

$\therefore\ \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \vec{b} + \vec{a}$

From (1) and (2) , we have

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

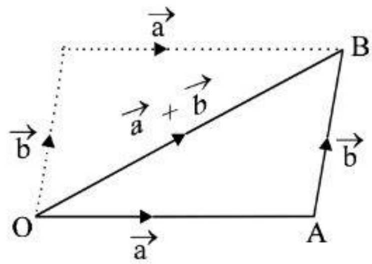


Fig.– 4

2. Vector Addition is Associative.

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.

Proof : Let the vectors $\vec{a}, \vec{b}, \vec{c}$ .be represented by the directed segments $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}$ respectively; so that *(fig.5)*

$\vec{a} = \overrightarrow{OA}, \ \vec{b} = \overrightarrow{AB}, \quad \vec{c} = \overrightarrow{BC}$

Then $\vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC})$

$= \overrightarrow{OA} + \overrightarrow{AC}$ [Δ Law of addition]

$= \overrightarrow{OC}$ [Δ Law of addition]

$\therefore \ \vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{OC}$ ....(1)

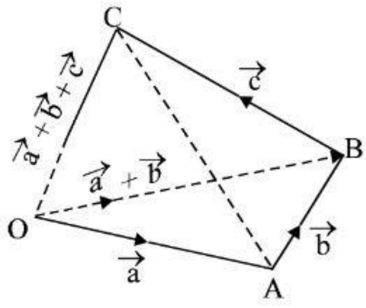


Fig. – 5

Again, $(\vec{a} + \vec{b}) + \vec{c} = (\overrightarrow{OA} + \overrightarrow{AB}) + \overrightarrow{BC}$

$= \overrightarrow{OB} + \overrightarrow{BC}$ [Δ Law of addition]

$= \overrightarrow{OC}$ [Δ Law of addition]

$\therefore \ (\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{OC}$ ....(2)

From (1) and (2), we get

$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

Remarks : The sum of three vectors $\vec{a}, \vec{b}, \vec{c}$ is independent of the order in which they are added and is written as $\vec{a} + \vec{b} + \vec{c}$.

(3) Existence of Additive Identity :

For any vector $\vec{a}$, $\vec{a} + \vec{O} = \vec{a}$, where $\vec{O}$ is a null (zero) vector.

Proof : Let the vector $\vec{a}$ be represented by the directed segment $\overrightarrow{OA}$; so that $\vec{a} = \overrightarrow{OA}$.

Also let the Zero Vector $\vec{O}$ be represented by the directed segment $\overrightarrow{AA}$;

So that $\vec{O} = \overrightarrow{AA}$

Then $\vec{a} + \vec{O} = \overrightarrow{OA} + \overrightarrow{AA}$

$= \overrightarrow{OA}$ [By Triangle law of addition]

$= \vec{a}$

Thus, $\vec{a} + \vec{O} = \vec{a}$

Note : In view of the above property, the null vector is called the additive identity.

Property 4 : Existence of Additive Inverse

For any vector $\vec{a}$, there exists another vector $-\vec{a}$ such that

$$\vec{a} + (-\vec{a}) = \vec{O}$$

Proof : Let $\overrightarrow{OA} = \vec{a}$, there exists another $\overrightarrow{AO} = -\vec{a}$

$\therefore \ \vec{a} + (-\vec{a}) = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = \vec{O}$ [By Δ Law]

Note : In view of the above property, the vector $(-\vec{a})$ is called the **additive inverse** of the vector $\vec{a}$.

Substraction of Vectors :

If $\vec{a}$ and $\vec{b}$ are two given vectors, then the substraction of $\vec{b}$ from $\vec{a}$ (denoted by $\vec{a}-\vec{b}$) is defined as addition of $-\vec{b}$ to $\vec{a}$.

i.e. $\vec{a}-\vec{b} = \vec{a}+(-\vec{b})$

$\therefore$ It is clear that

Multiplication of a Vector by a Scalar

If $\vec{a}$ is any given vector and m is any given scalar, then the product m $\vec{a}$ or $\vec{a}$ m of the vector $\vec{a}$ and the scalar m is a vector whose

(i) **Magnitude =** |m| times that of the vector $\vec{a}$.

In other words, $m\vec{a} = |m| \times |\vec{a}|$

$= m \times |\vec{a}|$ if $m \geq 0$

$= -m \times |\vec{a}|$ if $m < 0$

(ii) Support is same or parallel to that of the support of $\vec{a}$

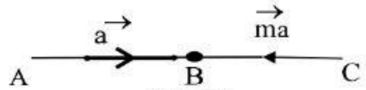
and (iii) Sense is same to that of $\vec{a}$ if $m > 0$ and opposite to that of $\vec{a}$ if $m < 0$.

Geometrical Representation :

Let the vector $\vec{a}$ be represented by the directed segment $\overrightarrow{AB}$

Case I . Let m > 0. Choose a point C and AB on the same side of A as B such that

$|\overrightarrow{AC}| = m\,|\overrightarrow{AB}|$, ***(fig.6)***



(m > 0)

Fig. – 6

Then the vector $\overrightarrow{ma}$ is represented by $\overrightarrow{AC}$.

Case II : Let m<0. Choose a point C on AB on the side of A opposite so that of B such that, ***(fig.7)***

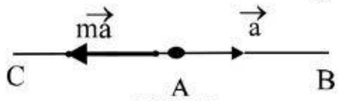


Fig. –7

$|\overrightarrow{AC}| = m\,, |\overrightarrow{AB}|$

Then the vector $\overrightarrow{ma}$ is represented by $\overrightarrow{AC}$.

Linearly Dependent and Independent Vectors

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be **linearly dependent** if there exists a scalar t ($\neq$0), such that $\vec{a} = t\vec{b}$

This can be the case if and only if the vectors $\vec{a}$ and $\vec{b}$ are parallel.

If the vectors $\vec{a}$ and $\vec{b}$ are not **linear dependent** they are said to be linearly independent and in this case $\vec{a}$ and $\vec{b}$ are not parallel vectors.

Thus, if $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$, then $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if A, B, C lie in a straight line; othewise they are linearly independent.

Properties of Multiplication of a Vector by a Scalar

(I) **Associative Law**

If $\vec{a}$ is any vector and m, n are any scalars, then $m(\overrightarrow{na}) = mn(\vec{a})$

Proof : If any one or more of m, n or a are zero, then $m(\overrightarrow{na}) = mn(\vec{a})$[$\because$ Each side = $\vec{0}$]

While if $m \neq 0, n \neq 0, \vec{a} \neq \vec{0}$, then the following four cases arise :

(i) $m > 0, n < 0$ (ii) $m < 0, n > 0$

(iii) $m > 0, n > 0$ (iv) $m < 0, n < 0$

Case (I) : When $m > 0, n < 0$

Let $\vec{a}$ be represented by the directed line segment $\overrightarrow{AB}$.*(fig .8)*

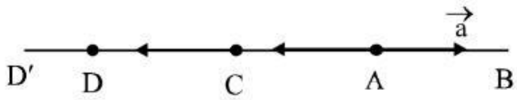


Fig. 8

Since $n < 0$, take a point C on AB on the side of A opposite to that of B such that $\overrightarrow{AC}$ represents $\overrightarrow{na}$

i.e.$|\overrightarrow{AC}| = |n|\,|\overrightarrow{AB}|$

Since $m > 0$, take point D on AB on the same side of A as C such $\overrightarrow{AD}$ represents $m(\overrightarrow{na})$

i.e. $|\overrightarrow{AD}| = m\,|\overrightarrow{AC}| = m|n|\,\overrightarrow{AB}|$......[1]

Again, since $m > 0, n < 0$, so that $mn < 0$; take a point D′ on AB on the side of A opposite to that of B such that AD′ represents $|mn|\,\vec{a}$.

i.e.,$|\overrightarrow{AD'}| = |mn|\,|\overrightarrow{AB}| = |m|n||\overrightarrow{AB}| = m\,|n|\,|\overrightarrow{AB}|$ [$\because |m| = m$ as $m > 0$]

$\therefore |\overrightarrow{AD'}| = m\,|n|\,|\overrightarrow{AB}|$ ......(2)

From (1) and (2), we get

$|\overrightarrow{AD'}| = |\overrightarrow{AD}|$

which shows that D and D′ coincide, proving that $m(\overrightarrow{na}) = (mn)\vec{a}$

Proceeding on the same lines, the other three cases can be similarly proved.

(2) Distributive Law : If m, n are any scalars and $\vec{a}$ is any vector, then

$(m + n)\vec{a} = \overrightarrow{ma} + \overrightarrow{na}$

Proof : If $\vec{a} = \vec{0}$ or m, n are both zero, then

$(m + n)\vec{a} = \overrightarrow{ma} + \overrightarrow{na}$ [$\because$ Each side = $\vec{0}$]

But if $\vec{a} \neq \vec{0}$, the following three cases arise :

(1) $m + n > 0$ (2) $m + n = 0$ and

(3) $m + n < 0$

Case - I . Here $m + n > 0$

The following sub-cases arise :

(i) $m > 0, n > 0$ (ii) $m > 0, n < 0$

and (iii) $m < 0, n > 0$

Let $\vec{a}$ be represented by the directed segment $\overrightarrow{AB}$.

Since m + n > 0, take a point C on AB on the same side of A as B such that $\overrightarrow{AC}$ represents $(m + n)\vec{a}$. *(fig. 9)*

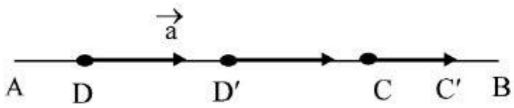


Fig. – 9

i.e. $|\overrightarrow{AC}| = (m + n)|\overrightarrow{AB}|$ ....(1)

Sub-case, (i) Since m > 0 , take a point D on the same side of A as B such that $\overrightarrow{AB}$ represent $\overrightarrow{ma}$.

i.e. $|\overrightarrow{AD}| = m|\overrightarrow{AB}|$ ....(2)

Again since n > 0, take a point D′ on the same side of A as B such that $\overrightarrow{AD'}$ represents $\overrightarrow{na}$.

i.e. $|\overrightarrow{AD'}| = n|\overrightarrow{AB}|$ ....(3)

Thus, $\overrightarrow{ma} + \overrightarrow{na}$ is represented by $\overrightarrow{AC'}$ (where C' is on the same side of A as B) such that

$|\overrightarrow{AC'}| = |\overrightarrow{AD}| + |\overrightarrow{AD'}|$

$= m|\overrightarrow{AB}| + n|\overrightarrow{AB}|$ [From (2) and (3)]

$\Rightarrow |\overrightarrow{AC'}| = (m + n)|\overrightarrow{AB}|$ ....(4)

$\therefore$ From (1) and (4), $|\overrightarrow{AC'}| = |\overrightarrow{AC}|$, which shows that C and C′ coincide, proving that

$(m + n)\vec{a} = \overrightarrow{ma} + \overrightarrow{na}$

The other sub-cases of case (1) may be similarly proved.

Proceeding in the same way, we can prove the result for case 2 and case 3 also.
case 2 and 3 also.

3. If $\vec{a}$ and $\vec{b}$ are any two vectors and m is a scalar, then

$m(\vec{a} + \vec{b}) = \overrightarrow{ma} + \overrightarrow{mb}$.*(fig.10)*

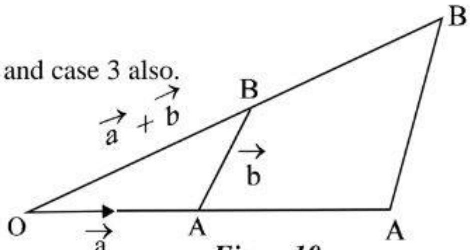


Fig. – 10

Position Vector of a Point

Let O be any point called the origin of reference or simple the origin. Let P be any other point.

Then $\overrightarrow{OP}$ is called the position vector of the point P relative to the point O.

Hence, with the choice of O as the origin of reference, a vector can be associated to every point P and conversely.*(fig .11)*

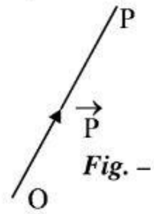


Fig. – 11

Representation of a vector in terms of the position vectors of its end points :

Let A and B be two given points and $\vec{a}, \vec{b}$ the position vectors of A, B respectively relative to a point O as the origin of reference; so that *(fig. 12)*

$\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

$\therefore$ From ΔOAB

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ [By Δ law of addition]

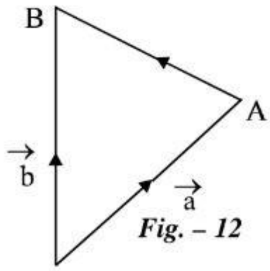


Fig. – 12

$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

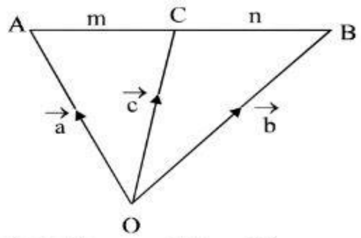
Note : $\overrightarrow{AB}$ = Position vector B – Position vector A.

SECTION FORMULA :

Statement . If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the point C which divides AB in the ratio m : n, where m and n are positive real numbers, has the position vector.

$$\vec{c} = \frac{n\vec{a} + m\vec{b}}{m+n}$$

Proof : Let O be the origin of reference and let $\vec{a}$ and $\vec{b}$ be the position vectors of the given points A and B so that *(fig.13)*



$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$

Let C divide AB in the ratio m : n

$\therefore \frac{AC}{CB} = \frac{m}{n}$ ....(i)

Hence $\frac{m}{n}$ is positive or negative according as C divides AB internally or externally. ***Fig. – 13***

We have to express the position vector $\overrightarrow{OC}$ of the point C in terms of those of A and B.

We re-write (1) as, nAC = mCB.

And obtain the vector equality $n\overrightarrow{AC} = m\overrightarrow{CB}$. Expressing the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of the position vectors of the end points, we obtain

$n(\overrightarrow{OC} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OC})$

$\Rightarrow (m+n)\overrightarrow{OC} = n\overrightarrow{OA} + m\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OC} = \frac{n\overrightarrow{OA} + m\overrightarrow{OB}}{m+n} = \frac{n\vec{a} + m\vec{b}}{m+n}$

Mid-point formula : If C is the mid-point of AB. then m : n = 1 : 1

$\therefore$ The position vector of c is given by $\overrightarrow{OC} = \frac{1\cdot\vec{a} + 1\cdot\vec{b}}{2}$

i.e. $\overrightarrow{OC} = \frac{\vec{a} + \vec{b}}{2}$

Hence the position vector of the mid point of the join of two points with position vectors, $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a} + \vec{b}}{2}$ or $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

Example – 1 : Prove that

(i) $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$ **(ii)** $|\vec{a}| - |\vec{b}| \leq |\vec{a} - \vec{b}|$ **(iii)** $|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$

Solution : (i) When A, B, C are not -collinear, draw a ΔABC such that, *(fig.14)*

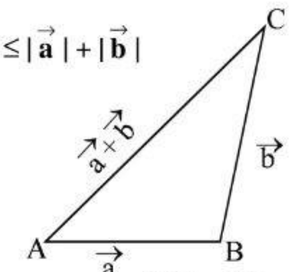


Fig. – 14

$\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$

Then $\vec{a} + \vec{b} = \overrightarrow{AC}$ [By Addition Law]

$\therefore$ AC < AB + BC (As sum of two sides is greater than the third side)

$\therefore |\overrightarrow{AC}| < |\overrightarrow{AB}| + |\overrightarrow{BC}|$

$|\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$ ....(1)

$[\because \overrightarrow{AB} = \vec{a},\ \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{AC} = \vec{a} + \vec{b}]$

When A, B and C are collinear, then, ***(fig.15)***

$\vec{a} = \overrightarrow{AB},\ \vec{b} = \overrightarrow{BC}$

$\vec{a} + \vec{b} = \overrightarrow{AC}$

Fig.– 15

$\because$ AC = AB + BC

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

$\Rightarrow |\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$

Combining (1) and (2), we get

$|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

(ii) $|\vec{a}| = |\vec{a} - \vec{b} + \vec{b}| = |(\vec{a} - \vec{b}) + \vec{b}|$...............(1)

But $|(\vec{a} - \vec{b}) + \vec{b}| \le |\vec{a} - \vec{b}| + |\vec{b}|$................(2)

From (1) and (2), we get

$|\vec{a}| \le |\vec{a} - \vec{b}| + |\vec{b}|$

$|\vec{a}| - |\vec{b}| \le |\vec{a} - \vec{b}|$

(iii) $|\vec{a} - \vec{b}| = |\vec{a} + (-\vec{b})| \le |\vec{a}| + |-\vec{b}|$

But $|-\vec{b}| = |\vec{b}|$

$\therefore |\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$.

Example – 2: **Prove by vector method that the lines segment joining the middle points of any two sides of a triangle is parallel to the third side and equal to half of it.**

Solution : Let ABC be a triangle in which D and E are the mid-points of AB and AC respectively. ***(fig.16)***

$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$

$= \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) = \frac{1}{2}\overrightarrow{BC}$

$\therefore$ DE || BC

Also, $DE = |\overrightarrow{DE}| = \left|\frac{1}{2}\overrightarrow{BC}\right| = \left|\frac{1}{2}\right| |\overrightarrow{BC}| = \frac{1}{2}BC$

Hence DE || BC and DE = $\frac{1}{2}$ BC.

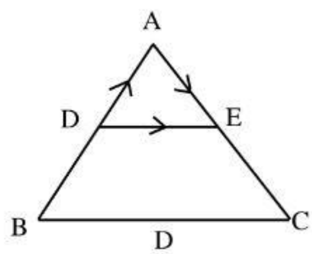


Fig. – 16

Components of a Vector in Two Dimensions

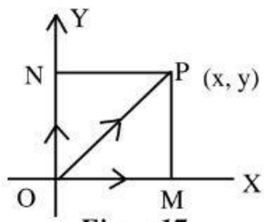


Fig. – 17

Let XOY be the co-ordinate plane let P(x, y) be a point in this plane. Join P. Draw $PM \perp OX$, and $PN \perp OY$.*(fig.17)*

Let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY.

Then $\vec{OM} = x\,\hat{i}$ and $\vec{ON} = y\,\hat{j}$.

$\vec{OM}$ and $\vec{ON}$ are called the vector components of $\vec{OP}$ along x-axis and y - axis respectively.

Thus the component of $\vec{OP}$ along x- axis is a vector x, whose magnitude is | x | and whose direction is along OX and OX′ according as x is positive or negative.

And, the component of $\vec{OP}$ along y - axis is a vector y, whose magnitude is | y | and whose direction is along OY or OY′ according as y is positive or negative.

$$\vec{OP} = \vec{OM} + \vec{MP} = \vec{OM} + \vec{ON} = x\,\hat{i} + y\,\hat{j}$$

Thus the position vector of the point P(x, y) is $x\,\hat{i} + y\,\hat{j}$

$OP^2 = OM^2 + MP^2 = x^2 + y^2$

$\Rightarrow OP = \sqrt{x^2 + y^2}$

$\therefore |\vec{OP}| = \sqrt{x^2 + y^2}$

Components of a vector along the co-ordinate axes.

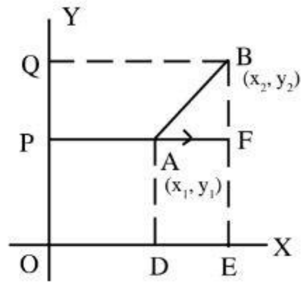


Fig. –18

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in XOY plane.

Draw $AD \perp OX$, *(fig.18)*

$BE \perp OX$ $AF \perp BE$, $AP \perp OY$ and $BQ \perp OY$

Clearly $AF = (x_2 - x_1)$

and $PQ = FB = (y_2 - y_1)$

Let $\hat{i}$ and $\hat{j}$ be unit vectors along x-axis and y-axis respectively.

Then $\overrightarrow{AF} = (x_2 - x_1)\,\hat{i}$

and $\overrightarrow{PQ} = \overrightarrow{FB} = (y_2 - y_1)\,\hat{j}$

Clearly $\overrightarrow{AB} = \overrightarrow{AF} + \overrightarrow{FB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$

Then component of $\vec{AB}$ along x - axis $= (x_2 - x_1)\,\hat{i}$

And component of $\vec{AB}$ along y - axis $= (y_2 - y_1)\,\hat{j}$

Also $|\overrightarrow{AB}| = AB = \sqrt{AF^2 + FB^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Components of Vector in three dimensions :

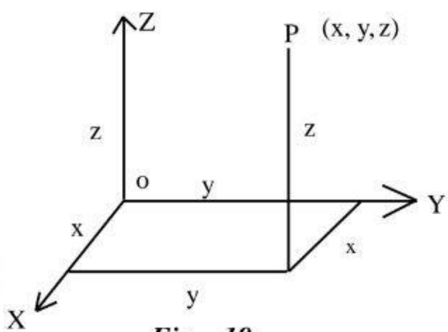


Fig. –19

Let OX, OY and OZ be three mutually perpendicular lines, taken as co-ordinate axis. Then the planes XOY, YOZ and ZOX are respectively known as XY plane, YZ plane and ZX plane. *(fig.19)*

Let P be any point in space. Then the distances of P from YZ- plane. ZX – plane and XY - plane are respectively called x-cordinate, y-cordinate and z-cordinate of P and we write P as P(x, y, z)

Position Vector of Point in space :

Let P(x, y, z) be a point in space with reference to three co-ordinate axes. OX, OY and OZ. Though P draw planes parallel to yz-plane zx-plane and xy-plane meeting the axes OX, OY and OZ at A, B and C respectively.

The OA = x, OB = y and OC = z

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vector along OX, OY and OZ respectively. *(fig. 20)*

Then $\overrightarrow{OA} = x\,\hat{i}$, $\overrightarrow{OB} = y\,\hat{j}$, $\overrightarrow{OC} = z\,\hat{k}$

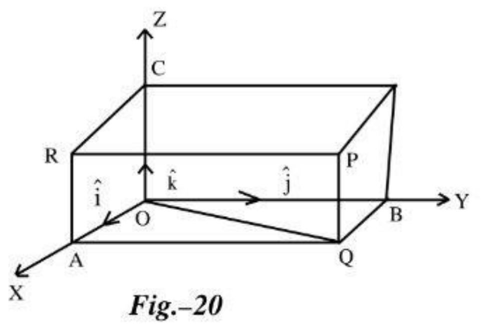


Fig.–20

Now $\overrightarrow{OP} = \overrightarrow{QP} + \overrightarrow{QP} = (\overrightarrow{OA} + \overrightarrow{AQ}) + \overrightarrow{QP}$

$(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$ $\left[\because \overrightarrow{AQ} = \overrightarrow{OB} \text{ and } \overrightarrow{QP} = \overrightarrow{OC}\right]$

$= x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

Thus, the position vector of a point

P (x, y, z) is the vector $(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$

Now $OP^2 = OQ^2 + QP^2 = (OA^2 + AQ^2) + QP^2$

$= (OA^2 + OB^2 + OC^2) = x^2 + y^2 + z^2$

$OP = \sqrt{x^2 + y^2 + z^2}$

$|\overrightarrow{OP}| = OP = \sqrt{x^2 + y^2 + z^2}$

If $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$,

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Components of Vector : If $\overrightarrow{OP}$ is the position vector of a point P(x, y, z) in space, then

$\overrightarrow{OP} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

The vectors $x\,\hat{i}$, $y\,\hat{j}$, $z\,\hat{k}$ are called the components of $\overrightarrow{OP}$ along x - axis y - axis and z-axis respectively.

ASSIGNMENTS

1. Show that the there points A(2, –1, 3), B (4, 3, 1) and C (3, 1, 2) are co-llinear.
2. Prove by vector method that the medians of a triangle are concurrent.
3. Find a unit vector in the direction of $(\vec{a} + \vec{b})$ where $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ & $\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$.

Scalar or Dot Product

Definition :

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ with magnitude a and b respectively, denoted by $\vec{a} \cdot \vec{b}$, is defined as the scalar ab cos θ, where θ is the angle between of $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$.

Thus $\vec{a} \,.\, \vec{b} = ab \cos \theta$.

Geometrical Meaning of Scalar Product

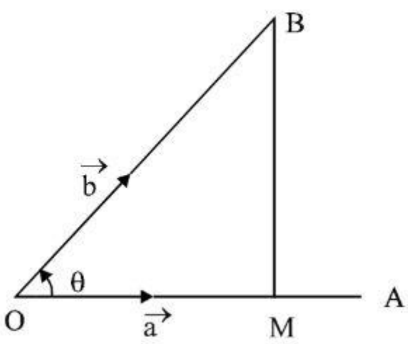


Fig. – 21

As we see in above figure that *(fig.21)*

$|OM| = |OB| \cos \theta = |\vec{b}| \cos \theta$ = projection of $\vec{b}$ on $\vec{a}$

$\therefore$ a.b= $|\vec{a}|\,(|\vec{b}| \cos \theta)$

= modulus of $\vec{a}$ × projection of $\vec{b}$ on $\vec{a}$ which gives,

projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

Similarly , if we drop a perpendicular from A on OB such that N is the foot of the perpendicular, then *(fig.22)*

ON = projection of $\vec{a}$ on $\vec{b}$ and ON = OA cos θ

$= |\vec{a}| \cos\theta$ = Now $\vec{a} \cdot \vec{b} = |\vec{b}|(|\vec{a}|\cos\theta)$

= magnitude of $\vec{b}$ × projection of $\vec{a}$ on $\vec{b}$ which gives that

projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

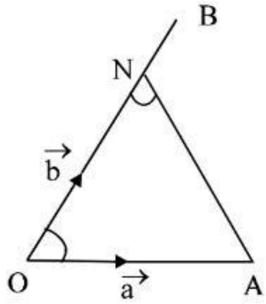


Fig. – 22

Thus we can conclude that

(i) The dot product of two vectors is equal to the magnitude of one vector multiplied by the projection of the other on it.

(ii) The (scalar) projection of one vector on another.

$= \dfrac{\text{Dot product of vectors}}{\text{Magnitude of the vector}}$ on which the projection is taken.

3. Commutative and distributive Properties of Scalar Product :

1. Scalar product of two vectors obeys commutative law i.e.,

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

2. Scalar product obeys distributive law i.e.

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

Other properties of scalar product : Apart from commutative and distributive properties. Scalar product has some ther properties as follow :

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2 \Rightarrow \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

2. $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a}$ is ⊥ to $\vec{b}$.

   Hence $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ and $\hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{i}\cdot\hat{k} = 0$

3. Scalar product in terms of components :

   If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

   then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

4. Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab} = \hat{a}\cdot\hat{b}$

   In terms of components $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

5. Projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ $\vec{a}, \hat{b}$ and projection of $\vec{b}$ on $\vec{a}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \hat{a}.\vec{b}$

6. $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$ or $(\vec{a}+\vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a}.\vec{b}$

7. $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$ or $(\vec{a}-\vec{b})^2 = \vec{a}^2 + \vec{b}^2 - 2\vec{a}.\vec{b}$

8. $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$ or $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = \vec{a}^2 - \vec{b}^2$

Components of a Vector $\vec{r}$ along and perpendicular to a given Vector $\vec{a}$ in the Plane of $\vec{a}$ and $\vec{r}$.

The resolved part of $\vec{r}$ in the direction of $\vec{a}$ = $\left(\dfrac{\vec{a}\cdot\vec{r}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$

The resolved part of r is perpendicular to a is $\left(\dfrac{\vec{r}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$

Example –1: Find the scalar and vector projections of $\hat{i}-\hat{j}-\hat{k}$ on $\hat{i}+\hat{j}+3\hat{k}$

Solution : Given $\vec{a}=\hat{i}-\hat{j}-\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-3\hat{k}$

Scalar projection of $\vec{a}$ on $\vec{b}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$=\frac{(\hat{i}-\hat{j}-\hat{k})(3\hat{i}+\hat{j}+3\hat{k})}{\sqrt{3^2+1^2+3^2}}=\frac{3-1-3}{\sqrt{19}}=\frac{-1}{\sqrt{19}}$$

Vector Projection of $\vec{a}$ on $\vec{b}=\left(\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\cdot\dfrac{\vec{b}}{|\vec{b}|}$

$$=\left(\frac{-1}{\sqrt{19}}\right)\cdot\left(\frac{3\hat{i}+j+3\hat{k}}{\sqrt{3^2+1^2+3^2}}\right)$$

$$=\frac{-1}{\sqrt{19}}\left(\frac{3\hat{i}+\hat{j}+3\hat{k}}{\sqrt{19}}\right)=-\left(\frac{3}{19}\hat{i}+\frac{\hat{j}}{19}+\frac{3\hat{k}}{19}\right)$$

ASSIGNMENTS

1. $\vec{a},\vec{b},\vec{c}$ are there rutually perpendicular vectors of the same magnitude prove that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined in the vectors $\vec{a},\vec{b}$ & $\vec{c}$.

2. Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a}=\hat{i}-\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+3\hat{k}$

3. Find the angle between the vector $\vec{a}=-\hat{i}+\hat{j}-2\hat{k}$ & $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$

Vector Product or Cross Product

The vector product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\times\vec{b}$ is defined as the vector $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|$ $\sin\theta$. $\hat{n}$ where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle from $\vec{a}$ and $\vec{b}$ such that $\vec{a}$ and $\vec{b}$ and $\hat{n}$ are the right handed system.

Angle between two vectors :

Let θ be the angle between $\vec{a}$ and $\vec{b}$. The $\vec{a}\times\vec{b}=(ab\sin\theta)\,\hat{n}$, where $|\vec{a}| = a$ and $\vec{b} = b$

$\therefore |\vec{a} \times \vec{b}| = (ab \sin\theta)\,|\hat{n}| = ab \sin\theta$

$[\because |\hat{n}| = 1]$

$ab \sin\theta = |\vec{a} \times \vec{b}|$

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{ab} = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\theta = \sin^{-1}\left\{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right\}$$

Unit vector perpendicular to two vectors :

Clearly $(\vec{a} \times \vec{b})$ is a vector, perpendicular to each one of the vector $\vec{a}$ and $\vec{b}$, so a unit vector $\hat{n}$ perpendicular to each one of the vector $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Properties of vector product :

(i) Vector product is not commutative

i.e. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(ii) For any vecrors $\vec{a}$ and $\vec{b}$ i.e. $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

(iii) For any scalar m prove that $(m\vec{a}) \times \vec{b} \Rightarrow m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$

(iv) For any vectors $\vec{a}, \vec{b}, \vec{c}$ present $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

(v) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ $\vec{a} \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})$

(vi) The vector product of two parallel or collinear vectors is zero.

(vii) For any vector $\vec{a}$ is $\vec{a} \times \vec{a} = \vec{0}$

(viii) If $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ are the parallel or collinear.

(ix) If the vectors $\vec{a}$ and $\vec{b}$ are parallel (or collinear) then $\theta = 0$ or $180°$, $\sin\theta = 0$

Vector product of orthonomal Triad of unit vectors :

Vector products of unit vectors $\hat{i}, \hat{j}, \hat{k}$ from a right-handed system of mutually perpendicular vectors. ***(fig.23)***

$\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$

$\hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times j$

$\hat{k} \times \hat{i} = j = -\hat{i} \times \hat{k}$

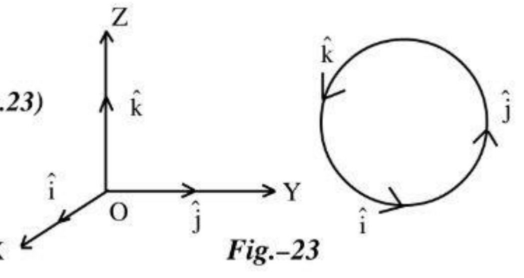


Fig.–23

Geometrical Interpretation of Vector Product or Cross Product

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Then $\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\,\hat{n}$

$= |\vec{a}|(|\vec{b}| \sin\theta)\,\hat{n} = |\vec{a}|\,|BM|\,\hat{n}$

Now $|\vec{a} \times \vec{b}| = |\vec{a}|\,|BM|$

= Area of the parallelogram with sides $\vec{a}$ and $\vec{b}$. ***(fig 24)***

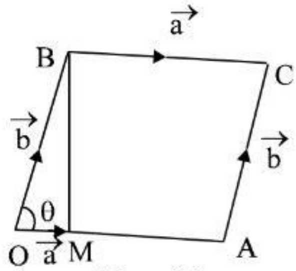


Fig. –24

Therefore, $\vec{a} \times \vec{b}$ is a vector whose magnitude is equal to area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.

From this it can be concluded that Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

Example – 1: Find the area of parallelogram whose adjacent sides are determined by the vectors.

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ **and** $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Solution : We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = (8\hat{i} - 10\hat{j} + 4\hat{k})$

$\therefore$ Required area $= |\vec{a} \times \vec{b}|$

$= \sqrt{8^2 + (-10)^2 + 4^2} = \sqrt{180} = 6\sqrt{5}$ sq. units

Example – 2: Find the area of a parallelogram whose diagonals are determined by the vectors.

$\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ **and** $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$

Solution : We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (-2\hat{i} - 14\hat{j} - 10\hat{k})$

$\therefore$ Required area $= \frac{1}{2}|\vec{a} \times \vec{b}|$

$= \frac{1}{2}\sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \frac{1}{2}\sqrt{300} = 5\sqrt{3}$ sq. units.

ASSIGNMENTS

1. Find the area of the triangle whose adjacant sides are $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ & $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$

2. Find a unit vector perpendicular to both the vector $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ & $\vec{b} = 3\hat{i} - \hat{j} + 3\hat{k}$

3. Find the angle between the vectors $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ & $\vec{b} = \hat{i} + 3\hat{j} + 2\hat{k}$

As the circle is symmetrically situated about both X −axis and Y −axis, the area of the circle is defined by,

$A = 4\int_0^a \sqrt{a^2 - x^2}\, dx$

$= 4\left|\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a$

$= 4\frac{a^2}{2}\sin^{-1}1 = 2a^2\frac{\pi}{2} = \pi a^2.$

CHAPTER - 5

DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS

DEFINITION:-An equation containing an independent variable (x), dependent variable (y) and differential co-efficients of dependent variable with respect to independent variable is called a differential equation.

For distance,

1. $\frac{dy}{dx} = \sin x + \cos x$
2. $\frac{dy}{dx} + 2xy = x^3$
3. $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Are examples of differential equations.

ORDER OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest order derivative appearing in the equation.

Example:-In the equation, $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$,

The order of highest order derivative is 2. So, it is a differential equation of order 2.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the integral power of the highest order derivative occurring in the differential equation, after the equation has been expressed in a form free from radicals and fractions.

Example:-Consider the differential equation $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$

In this equation the power of highest order derivative is 1.So, it is a differential equation of degree 1.

Example:-Find the order and degree of the differential equation

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = K\frac{d^2y}{dx^2}$$

Solution:- By squaring both sides, the given differential equation can be written as $K^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$

The order of highest order derivative is 2.So, its order is 2.
Also, the power of the highest order derivative is 2.So, its degree is 2.

FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed by eliminating certain arbitrary constants from a relation in the independent variable, dependent variable and constants.

Example:-Form the differential equation of the family of curves $y = a\sin(bx + c)$, a and c being parameters.

Solution:-We have $y = a\sin(bx + c)$ -------------(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = ab\cos(bx + c) \quad \text{----------------------(2)}$$

Differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -ab^2\sin(bx + c) \quad \text{-------------------(3)}$$

Using (1) and (3), we get

$$\frac{d^2y}{dx^2} = -b^2y$$

$$\therefore \quad \frac{d^2y}{dx^2} + b^2y = 0$$

This is the required differential equation.

Example:-Form the differential equation by eliminating the arbitrary constant in $y = Atan^{-1}x$.

Solution:-We have, $y = Atan^{-1}x$ ------------------(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{A}{1+x^2} \quad \text{-------------------(2)}$$

Using (1) and (2), we get

$$\frac{dy}{dx} = \frac{y}{(1+x^2)tan^{-1}x}$$

$$\therefore \ (1 + x^2)tan^{-1}x\frac{dy}{dx} = y$$

This is the required differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is a relation $(like\ y = f(x)\ or\ f(x,y) = 0)$between the variables which satisfies the given differential equation.

GENERAL SOLUTION

The general solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.

PARTICULAR SOLUTION

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

SOLUTION OF FIRST ORDDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

We shall discuss some special methods to obtain the general solution of a first order and first degree differential equation.

1. Separation of variables

2. Linear Differential Equations

3. Exact Differential Equations

SEPARATION OF VARIABLES

If in a first order and first degree differential equation, it is possible to separate all functions of x and dx on one side, and all functions of y and dy on the other side of the equation, then the variables are said to be separable.Thus the general form of such an equation is $f(y)dy = g(x)dx$
Then, Integrating both sides, we get

$\int f(y)dy = \int g(x)dx + C$ as its solution.

Example:-Obtain the general solution of the differential equation

$$9y\frac{dy}{dx} + 4x = 0$$

Solution:- We have, $9y\frac{dy}{dx} + 4x = 0$

$$\Rightarrow 9y\frac{dy}{dx} = -4x$$

$$\Rightarrow 9y\,dy = -4x\,dx$$

Integrating both sides, we get

$$9\int ydy = -4\int xdx$$

$$\Rightarrow \frac{9}{2}\cdot y^2 = \frac{-4}{2}x^2 + K$$

$$\Rightarrow 9y^2 = -4x^2 + C \qquad \text{(Where C=2K)}$$

$$\Rightarrow 4x^2 + 9y^2 = C$$

This is the required solution

LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear, if the dependent variable and its differential co-efficients occurring in the equation are of first degree only and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q, \quad \text{-----------------(1)}$$

Where P and Q are functions of x.

To solve linear differential equation of the form (1),
at first find the Integrating factor = $e^{\int P\,dx}$ --------------------(2)

It is important to remember that

$$\text{I.F} = e^{\int P\cdot dx}$$

Then, the general solution of the differential equation (1) is

$$y.(I.F) = \int Q.(I.F)dx + C \quad \text{-------------------(3)}$$

Example:-Solve $\frac{dy}{dx} + y\sec x = \tan x$

Solution:-The given differential equation is

$$\frac{dy}{dx} + (\sec x)y = \tan x \quad \text{-----------------(1)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \text{ , where } P = \sec x \text{ and } Q = \tan x$$

$\therefore$ I.F= $e^{\int P\cdot dx} = e^{\int \sec x dx} = e^{\ln(\sec x + \tan x)}$

So, I.F=$\sec x + \tan x$

$\therefore$ The general solution of the equation (1) is

$$y.(I.F) = \int Q(I.F)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \int (\tan x\ \sec x + tan^2 x)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \int (\tan x\ \sec x + sec^2 x - 1)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \sec x + \tan x - x + C$$

This is the required solution.

Example:-Solve:$(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

Solution:-The given differential equation can be written as

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2$$

$$\Rightarrow \frac{dy}{dx}+\frac{2x}{1+x^2}\cdot y=\frac{4x^2}{1+x^2} \quad \text{----------------(1)}$$

This is a linear equation of the form $\frac{dy}{dx}+Py=Q$,

Where $P=\frac{2x}{1+x^2}$ and $Q=\frac{4x^2}{1+x^2}$

We have, I.F = $e^{\int P\cdot dx}=e^{\int 2x/(1+x^2)dx}=e^{\ln(1+x^2)}=1+x^2$ ---------------(2)

∴The general solution of the given differential equation (1) is

$$y.(I.F)=\int Q.(I.F)dx+C$$

$$\Rightarrow \quad y(1+x^2)=\int\frac{4x^2}{1+x^2}.(1+x^2)dx+C$$

$$\Rightarrow \quad y(1+x^2)=4\int x^2dx+C$$

$$\Rightarrow \quad y(1+x^2)=\frac{4}{3}x^3+C$$

This is the required solution

EXACT DIFFERENTIAL EQUATIONS

DEFINITION:- A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

METHOD OF SOLUTION:-

The general solution of an exact differentia equation $Mdx+Ndy=0$ is

$$\int\limits_{(y=\text{constant})} Mdx+\int(terms\ of\ N\ not\ containing\ x)dy=C,$$

Provided $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Example:- Solve;$(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$.

Solution:-The given differential equation is of the form $Mdx+Ndy=0$.

Where, $M=x^2-4xy-2y^2$ and $N=y^2-4xy-2x^2$

We have $\frac{\partial M}{\partial y}=-4x-4y=\frac{\partial N}{\partial x}$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, so the given differential equation is exact.

∴ The general solution of the given exact differential equation is

$$\int\limits_{(y=\text{constant})} Mdx+\int(terms\ of\ N\ free\ from\ x)dy=C$$

$$\Rightarrow \quad \int\limits_{(y=\text{constant})}(x^2-4xy-2y^2)dx+\int y^2dy=C$$

$\Rightarrow \quad \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = C$
$\Rightarrow \quad x^3 - 6x^2y - 6xy^2 + y^3 = C.$
This is the required solution.

Example:- Solve;$(x^2 - ay)dx = (ax - y^2)dy$

Solution:-The given differential equation can be written as

$(x^2 - ay)dx + (y^2 - ax)dy = 0$ ----------------(1)
Which is of the form $Mdx + Ndy = 0$,
Where,$M = x^2 - ay$ and $N = y^2 - ax$.
We have $\frac{\partial M}{\partial y} = -a \text{ and } \frac{\partial N}{\partial x} = -a$
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation (1) is exact.
∴ The solution of (1) is $\int (x^2 - ay)dx + \int y^2 dy = C$
(y=constant)
$\Rightarrow \frac{x^3}{3} - axy + \frac{y^3}{3} = C$
$\Rightarrow x^3 - 3axy + y^3 = C,$
Which is the required solution.

Example:- Solve; $ye^{xy}dx + (xe^{xy} + 2y)dy = 0.$

Solution:- The given differential equation is $ye^{xy}dx + (xe^{xy} + 2y)dy = 0,$

Which is of the form $Mdx + Ndy = 0.$
Where,$M = ye^{xy}$ and $N = xe^{xy} + 2y$
We have $\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$
So the given equation is exact and its solution is
$\int ye^{xy}dx + \int 2ydy = C.$
(y=constant)
$\Rightarrow e^{xy} + y^2 = C$

Example:- Solve; $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$

Solution:- The given equation is of the form $Mdx + Ndy = 0,$
Where,$M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$
We have $\frac{\partial M}{\partial y} = 12xy = \frac{\partial N}{\partial x}$.
So the given equation is exact and its solution is
$\int (3x^2 + 6xy^2)dx + \int (4y^3)dy = C$
(y=constant)
$\Rightarrow \frac{3x^3}{3} + \frac{6}{2}x^2y^2 + \frac{4}{4}y^4 = C$

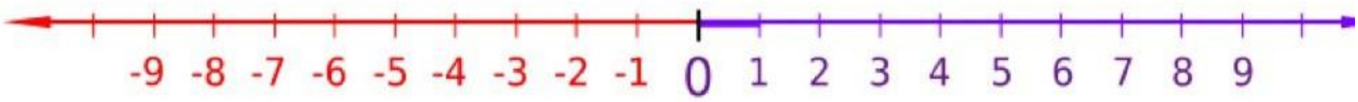
$\Rightarrow x^3 + 3x^2y^2 + y^4 = C$
This is the required solution.

Co-Ordinate System

In geometry, a **coordinate system** is a system which uses one or more numbers, or **coordinates**, to uniquely determine the position of a point. The order of the coordinates is significant and they are sometimes identified by their position in an ordered tuple and sometimes by a letter, as in "the *x* coordinate".

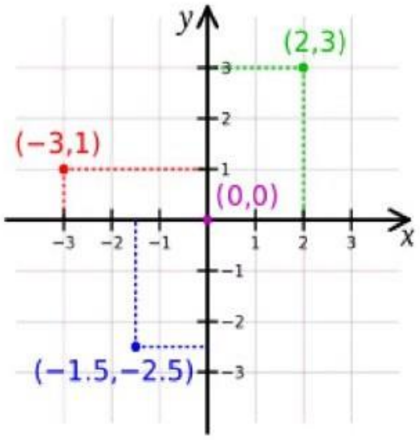
Number Line

The simplest example of a coordinate system is the identification of points on a line with real numbers using the *number line*. In this system, an arbitrary point *O* (the*origin*) is chosen on a given line. The coordinate of a point *P* is defined as the signed distance from *O* to *P*, where the signed distance is the distance taken as positive or negative depending on which side of the line *P* lies. Each point is given a unique coordinate and each real number is the coordinate of a unique point.[4]



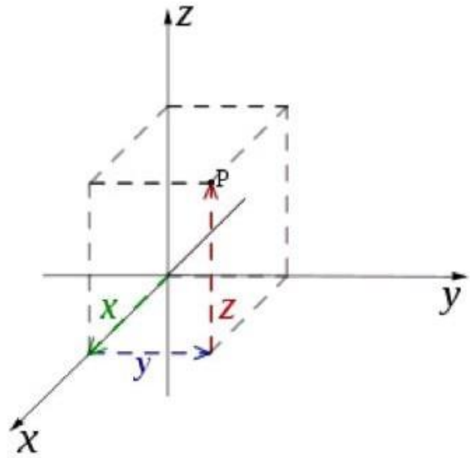
Cartesian Co-ordinate System

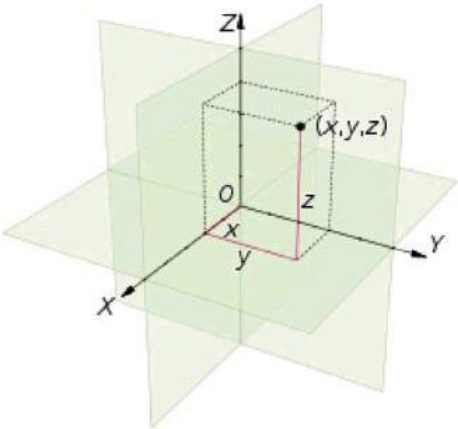
In the plane, two perpendicular lines are chosen and the coordinates of a point are taken to be the signed distances to the lines.



Three Dimension

In three dimensions, three perpendicular planes are chosen and the three coordinates of a point are the signed distances to each of the planes.





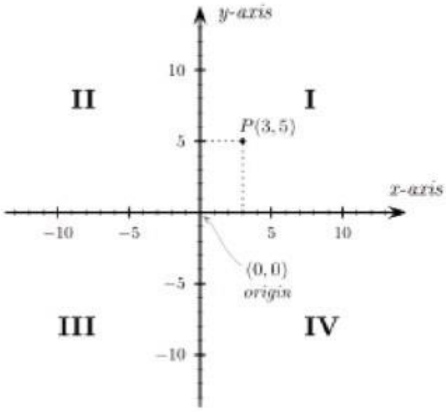
Choosing a Cartesian coordinate system for a three-dimensional space means choosing an ordered triplet of lines (axes) that are pair-wise perpendicular, have a single unit of length for all three axes and have an orientation for each axis. As in the two-dimensional case, each axis becomes a number line. The coordinates of a point *P* are obtained by drawing a line through *P* perpendicular to each coordinate axis, and reading the points where these lines meet the axes as three numbers of these number lines.

Alternatively, the coordinates of a point *P* can also be taken as the (signed) distances from *P* to the three planes defined by the three axes. If the axes are named *x*, *y*, and *z*, then the *x*-coordinate is the distance from the plane defined by the *y*and *z* axes. The distance is to be taken with the + or − sign, depending on which of the two half-spaces separated by that plane contains *P*. The *y* and *z* coordinates can be obtained in the same way from the *x*–*z* and *x*–*y* planes respectively.

The Cartesian coordinates of a point are usually written in parentheses and separated by commas, as in (10, 5) or(3, 5, 7). The origin is often labelled with the capital letter *O*. In analytic geometry, unknown or generic coordinates are often denoted by the letters *x* and *y* on the plane, and *x*, *y*, and *z* in three-dimensional space.

The axes of a two-dimensional Cartesian system divide the plane into four infinite regions, called **quadrants**, each bounded by two half-axes.

Similarly, a three-dimensional Cartesian system defines a division of space into eight regions or **octants**, according to the signs of the coordinates of the points. The convention used for naming a specific octant is to list its signs, e.g. (+ + +) or (− + −).



Distance between two points

The distance between two points of the plane with Cartesian coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is the Cartesian version of Pythagoras' theorem. In three-dimensional space, the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

which can be obtained by two consecutive applications of Pythagoras' theorem.

Example :

Prive that the point A(-1,6,6),B(-4,9,6),C(0,7,10) form the vertices of a right angled tringled.

Solution :

By distance formula

$AB^2 = (-4+1)^2 + (9-6)^2 + (6-6)^2 \ = 9 + 9 = 18$

$BC^2 = (0+4)^2 + (7-9)^2 + (10-6)^2 \ = 16 + 4 + 16 = 36$

$AC^2 = (0+1)^2 + (7-6)^2 + (10-6)^2 \ = 1 + 1 + 16 = 18$

Which gives $AB^2 + AC^2 = 18 + 18 = 36$

Hence ABC is a right angled isosceles triangle

Derivation Of Distance Formula

Fig

The distance between the point P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) is given by

PQ = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Proof:

Let $\overline{P`Q`}$ be the projection of $\overline{PQ}$ on the XY plane. $\overline{PP`}$ and . $\overline{QQ`}$ are parallel. So $\overline{PP`}$ and $\overline{QQ`}$ are co-planar. And $\overline{PP`QQ`}$ is a plane quadrilateral.

Let R be a point on $\overleftrightarrow{QQ`}$ so that $\overline{PR} \parallel \overline{P`Q`}$.

Since $\overleftrightarrow{P`Q`}$ lies on the XY plane and $\overleftrightarrow{PP`}$ is perpendicular to this plane , it follow from the definition of perpendicular geometry to a plane that $\overline{PP`}$ perpendicular $\overline{P`Q`}$. Similarly $\overline{QQ`}$ perpendicular to $\overline{PP`}$. $\overline{PR}$ being parallel to $\overline{P`Q`}$. It follow from plane geometry that PP`Q`R is a rectangle. So PR = P`Q`. and

< PRQ is a right angle.

P` and Q` being the projection of point P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) on the XY plane, they are given by P`$(x_1, y_1, 0)$ and Q$(x_2, y_2, 0)$. Therefore by the distance formula in the geometry of R^2.

P`Q` = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

In the rectangle PP`Q`R

P`P = Q`R

Therefore QR = | $z_2 - z_1$ |

In the right angled tringle PRQ , $PQ^2 = PR^2 + RQ^2$

= $(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

PQ = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Derive the division formula

Fig

If R(x,y,z) divides the segment joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) internally in ratio m : n ie $\frac{PR}{QR} = \frac{m}{n}$,then x = $\frac{mx_2+nx_1}{m+n}$, y = $\frac{my_2+ny_1}{m+n}$ and z = $\frac{mz_2+nz_1}{m+n}$

Proof :

Let P`, Q` and R` be the feet of the perpendicular from P, Q, R on the xy plane. Being perpendicular on the same plane $\overleftrightarrow{PP`}\ \overleftrightarrow{QQ`}\ \overleftrightarrow{RR`}$ are parallel lines. Since these parallel lines have a common transversal $\overleftrightarrow{PQ}$ they are co-planar. Let M and N be points on $\overleftrightarrow{RR`}$ and $\overleftrightarrow{QQ`}$ such that $\overleftrightarrow{PM}$

perpendicular $\overleftrightarrow{RR`}$ and $\overleftrightarrow{RN}$ perpendicular $\overleftrightarrow{QQ`}$. Since P`,R` and Q` are common to the xy-plane and plane of $\overleftrightarrow{PP`}\ \overleftrightarrow{QQ`}\ \overleftrightarrow{RR`}$ they must collinear because two plane intersect along a line.

It follows from the definition of the perpendicular to a plane that $< PP`R`, < RR`Q`\ and\ < QQ`R`$ are all right angles. It now follows from plane geometry that PP`R`M and RR`Q`N are rectangles. Also triangles RPM and QRN are similar

Hence $\frac{m}{n} = \frac{PR}{RQ} = \frac{PM}{RN} = \frac{P`R`}{R`Q`}$

($\therefore PM = P`R`\ and\ RN = R`Q`\ in\ the\ corresponding\ rectangle$)

Thus the point R` divides the segment $\overline{P`Q`}$ internally in the ration m : n.

P`, R` and Q` being projection of P(x_1, y_1, z_1) , R(x,y,z) and Q(x_2, y_2, z_2) on the xy plane have co-ordinate respectively $(x_1, y_1, 0)$,(x,y,0), $(x_2, y_2, 0)$.

If we restrict our consideration to the xy plane only we can regard the point P`,R`,Q` as having coordinate (x_1, y_1) ,(x,y), (x_2, y_2) .

Thus on the xy-plane the point R`(x,y) divides the segment joining P(x_1, y_1) and Q(x_2, y_2) internally in the ratio given by x = $\frac{mx_2+nx_1}{m+n}$, y = $\frac{my_2+ny_1}{m+n}$

Simillarly considering projection of P,Q,R on on another co-ordinate plane say YZ plane we can prove y = $\frac{my_2+ny_1}{m+n}$ and z = $\frac{mz_2+nz_1}{m+n}$

Thus we have x = $\frac{mx_2+nx_1}{m+n}$, y = $\frac{my_2+ny_1}{m+n}$ and z = $\frac{mz_2+nz_1}{m+n}$

External Division Formula

If R(x,y,z) devides the segment $\overline{PQ}$ joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) externally in ratio m : n ie $\frac{PR}{QR} = \frac{m}{n}$ then x = $\frac{mx_2-nx_1}{m-n}$, y = $\frac{my_2-ny_1}{m-n}$ and z = $\frac{mz_2-nz_1}{m-n}$

Example :

Find the ratio in which the line segment joining points (4,3,2) and (1,2,-3) is divided by the co-ordinate planes.

Solution :

Let the given points be denoted by A(4,3,2) and B(1,2,-3). If Q is the point where the line through A and B is met by xy-plane, then the co-ordinate of Q are $(\frac{k+4}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+2}{k+1})$, since Q devides $\overline{AB}$ in a ratio k:1 for some real value k. But being a point on the xy-plane, its z co-ordinate is zero.

Hence $\frac{-3k+2}{k+1} = 0 \;\; or\; k = \frac{2}{3}$

Similarly $\overline{AB}$ meets the xy-plane has its y-co-ordinate zero. Hence equating the y-co-ordinate to zero we get

$\frac{2k+3}{k+1} = 0 \;\; or\; k = -\frac{3}{2}$ ie the xz plane devides in a ratio 3:2. Equating x co-ordinate to zero we get $\frac{k+4}{k+1}$

k = -4.

Ie yz-plane divides $\overline{AB}$ externally in a ratio 4:1.

Direction Cosine and Direction Ratio

Fig

Let L be a line in space. Consider a ray R parallel to L with vortex at origin. (R can be taken as either $\overrightarrow{OP}$ or $\overrightarrow{OP`}$). let α, β, γ be the inclination between the ray R and $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ respectively.Them we define the direction cosine of L as $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Usually direction cosine of a line are denoted as <l,m,n>. for the above line l = $\cos\alpha$, m = $\cos\beta$,

n = $\cos\gamma$.

In the definition of the direction cosine of L the ray can be either $\overrightarrow{OP}$ or $\overrightarrow{OP`}$. Therefore if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosine of L then $\cos(\pi-\alpha)$, $\cos(\pi-\beta)$, $\cos(\pi-\gamma)$ can also be considered as direction cosine of L . The two set of direction cosine corresponds to the two opposite direction of a line L.

The direction cosine of the ray $\underset{OP}{\rightarrow}$ are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and of the ray $\underset{OP`}{\rightarrow}$ are $\cos(\pi-\alpha)$, $\cos(\pi-\beta)$, $\cos(\pi-\gamma)$.

Property Of Direction Cosine

A. Let O bethe origin and direction cosine of $\underset{OP}{\rightarrow}$ be l,m,n. If OP = r and P has a co-ordinate (x,y,z) then

   x = lr, y = mr, z = nr.

B. If l,m,n are direction cosines of a line then
   $l^2 + m^2 + n^2 = 1$

Direction Ratio

Let l,m,n be the direction cosine of the line such that none of the direction cosine is zero.

If a,b,c are non zero real number such that $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$ then a,b,c are the direction ratio of the line

Exceptionalcases:

1. If one of the direction cosine of a line L , say l = 0 and m $\neq$ 0, n $\neq$ 0 then direction ration of L are given by (0,b,c) where $\frac{b}{m} = \frac{c}{n}$ abd b and c are nonzero real number.
2. If two direction cosine are zero l = m = 0 and n $\neq$ 0 then obviously n = $\pm$1 and the direction ratio are (0,0,c) , c $\in$ R, c $\neq$ 0.

Finding Direction Cosine from Direction Ratio

If a,b,c are direction ration of a line then its direction cosine are given by

l = $\frac{a}{\pm\sqrt{a^2+b^2+c^2}}$, m = $\frac{b}{\pm\sqrt{a^2+b^2+c^2}}$, n = $\frac{c}{\pm\sqrt{a^2+b^2+c^2}}$

Direction Ratio of the line segment joining two points : $\frac{(x_2-x_1)}{\cos\alpha} = \frac{(y_2-y_1)}{\cos\beta} = \frac{(z_2-z_1)}{\cos\gamma}$

Angle between two lines with given Direction ratio

If L_1 *and* L_2 are not parallel lines having direction cosine $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$ and θ is the measure of angle between them then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

Proof :

Consider the ray $\underset{OP}{\rightarrow}$ and $\underset{OQ}{\rightarrow}$ such that $\underset{OP}{\rightarrow} \parallel L_1$ and $\underset{OQ}{\rightarrow} \parallel L_2$. $\underset{OP}{\rightarrow}$ and $\underset{OQ}{\rightarrow}$ are taken in such a way that m<POQ = θ and direction cosine of $\underset{OP}{\rightarrow}$ and $\underset{OQ}{\rightarrow}$ are respectively $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$. Let P andQ have a co-ordinate respectively (x_1, y_1, z_1) and (x_2, y_2, z_2)

In ΔOPQ $\cos\theta = \frac{OP^2+OQ^2-PQ^2}{2\,OP\,.\,OQ}$

$= \frac{(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)-\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}{2\ OP\,.\,OQ}$

$= \frac{x_1x_2+y_1y_2+z_1z_2}{OP\,.\,OQ} = \frac{x_1}{OP}\ \frac{x_2}{OQ} + \frac{y_1}{OP}\ \frac{y_2}{OQ} + \frac{z_1}{OP}\ \frac{z_2}{OQ} = l_1l_2 + m_1m_2 + n_1n_2$

Note that L_1 *and* L_2 is perpendicular then $\cos\theta = 0$.

1. Thus the line with direction cosine $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$ are perpendicular only if $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
2. If $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ are direction ratio of L_1 *and* L_2 and θ measures the angle between them then
   $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\pm\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$

Lines with direction ratio $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ are perpendicular if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

3. Since parallel lines have same direction cosine it follows from the definition of direction ratio that lines with direction ratio $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ are parallel if and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Example :

Find the direction cosine of the line which is perpendicular to the lines whose direction ratios are <1,-2,3> and <2,2,1>

Solution :

Let l, m, n be the direction cosine of the line which is perpendicular to the given lines. Then we have

l.1+m.(-2)+n.3 = 0 and l.2+m.2+n.1 =0

By cross multiplication we have

$$\frac{l}{-2-6} = \frac{m}{6-1} = \frac{n}{2+4}$$

Or, $\frac{l}{-8} = \frac{m}{5} = \frac{n}{6} = k \quad then\ l = -8k;\ m = 5k; n = 6k$

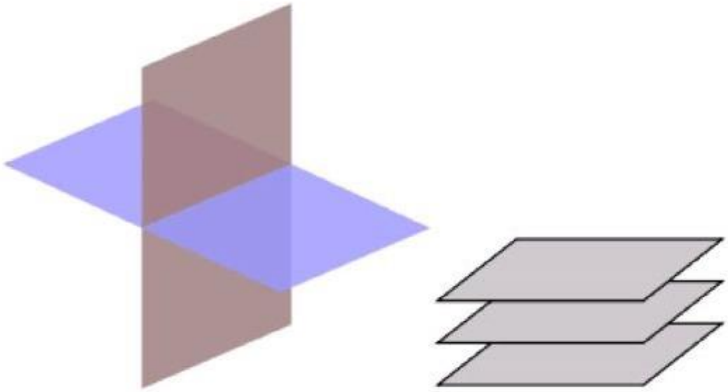
$l^2 + m^2 + n^2 = 1 \quad => (64 + 25 + 36)k^2 = 1$

Or $k^2 = \frac{1}{125} \quad => k = \frac{1}{5\sqrt{5}}$

$\therefore l = \frac{8}{5\sqrt{5}}; m = \frac{1}{\sqrt{5}}\ ; n = \frac{6}{5\sqrt{5}}$

Plane

In mathematics, a **plane** is a flat, two-dimensional surface. A plane is the two-dimensional analogue of a point (zero-dimensions), a line (one-dimension) and a solid (three-dimensions). Planes can arise as subspaces of some higher-dimensional space, as with the walls of a room, or they may enjoy an independent existence in their own right,



Properties

The following statements hold in three-dimensional Euclidean space but not in higher dimensions, though they have higher-dimensional analogues:

- Two planes are either parallel or they intersect in a line.
- A line is either parallel to a plane, intersects it at a single point, or is contained in the plane.
- Two lines perpendicular to the same plane must be parallel to each other.
- Two planes perpendicular to the same line must be parallel to each other.

Point-normal form and general form of the equation of a plane

In a manner analogous to the way lines in a two-dimensional space are described using a point-slope form for their equations, planes in a three dimensional space have a natural description using a point in the plane and a vector (the normal vector) to indicate its "inclination".

Specifically, let $\mathbf{r}_0$ be the position vector of some point $P_0 = (x_0, y_0, z_0)$, and let $\mathbf{n} = (a, b, c)$ be a nonzero vector. The plane determined by this point and vector consists of those points P, with position vector $\mathbf{r}$, such that the vector drawn from P_0 to P is perpendicular to $\mathbf{n}$. Recalling that two vectors are perpendicular if and only if their dot product is zero, it follows that the desired plane can be described as the set of all points $\mathbf{r}$ such that

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

(The dot here means a dot product, not scalar multiplication.) Expanded this becomes

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which is the *point-normal* form of the equation of a plane.[3] This is just a linear equation:

$$ax + by + cz + d = 0, \text{ where } d = -(ax_0 + by_0 + cz_0).$$

Conversely, it is easily shown that if a, b, c and d are constants and a, b, and c are not all zero, then the graph of the equation

$$ax + by + cz + d = 0,$$

is a plane having the vector $\mathbf{n} = (a, b, c)$ as a normal.[4] This familiar equation for a plane is called the *general form* of the equation of the plane.[5]

Example :

Find the equation of the plane through the point (1,3,4), (2,1,-1) and (1,-4,3).

Ans :

Any plane passing through (1,3,4) is given by

A(x-1) + B(y-3) + C(z-4) = 0 ….(1)

Where A,B,C are direction ratio of the normal to the plane.

Since the passes through(2,1,-1) and (1,-4,3) we have

A(2-1) +B(1-3)+C(-1-4) = 0

Or A -2B-5C = 0 …..(1)

A(1-1)+B(-4-3)+C(3-4) = 0

Or A.0 + B(-7)+C(-1) = 0

Or -7B-C = 0 …..(2)

By Type equation here.cross multiplication we get

$$\frac{A}{(-2)(-1)-(-5)(-7)} = \frac{B}{(-5)0-(1)(-1)} = \frac{C}{1(-7)-0.(-2)}$$

Or, $\frac{A}{-33} = \frac{B}{1} = \frac{C}{-7}$

Hence the direction ratio of the normal to the plane are 33,-1,7 and putting these values in (1), the equation of the required plane is

33(x-1)-1(y-3)+7(z-4) = 0

Or 33x – y +7z-58 = 0

Equation Of plane in normal form

Fig

Let p be the length of the perpendicular $\overline{ON}$ from the origin on the plane and let <l,m,n> be its direction cosines. Then the co-ordinate of the foot of the perpendicular N are (lp,mp,np).

If P(x,y,z) be any point on the plane then the direction ratio of $\overline{NP}$ are (x-lp,y-mp,z-np). Since $\overline{ON}$ is perpendicular to the plane it is also perpendicular to $\bar{N}P$

Hence

L(x – lp) + m(y – mp)+n(z –np) = 0

Or, lx +my+nz = $(l^2 + m^2 + n^2)p$

Or lx+my+nz =p

Example :

Obtain the normal form of equation of the plane 3x+2y+6z+1 = 0 and find the direction cosine and length of the perpendicular from the origin to this plane.

Solution :

The direction ratios of the normal to the plane are <3,2,6> and hence the direction cosines are

$$< \frac{3}{\pm\sqrt{9+4+36}}, \frac{2}{\pm\sqrt{9+4+36}}, \frac{6}{\pm\sqrt{9+4+36}} >$$

Length of the perpendicular from origin is

$$P = \frac{-D}{\pm\sqrt{A^2+B^2+C^2}} = \frac{-1}{\pm\sqrt{9+4+36}} = \frac{1}{7}$$

$(\therefore D \text{ is positive we choose negative before the radical sign to make } p > 0$

The equation of plane in normal form is

$$\frac{A}{-\sqrt{A^2+B^2+C^2}}\,x + \frac{B}{-\sqrt{A^2+B^2+C^2}}\,y + \frac{C}{-\sqrt{A^2+B^2+C^2}}\,z + \frac{D}{-\sqrt{A^2+B^2+C^2}} = 0$$

Or $\frac{3}{-7}\, x + \frac{2}{-7}\, y + \frac{6}{-7}\, z + \frac{1}{-7} = 0$

Distance Of a point from a plane

Fig

Let P(x_1, y_1, z_1) be a given point and Ax+By+Cz+D = 0 be the equation of a given plane. Draw $\overline{QN}$ normal to the plane at Q and $\overline{PM}$ perpenticular to $\overline{QN}$. Join $\overline{PQ}$ .If R be the foot of the perpentcular drawn from the point Pto the given plane, then

D= PR=QM= projection of $\overline{PQ}$ on $\overrightarrow{QN}$. $\overrightarrow{QN}$ being normal to the given plane Ax+By+Cz+D = 0 the direction ratio of $\overrightarrow{QN}$ are <A,B,C> and the direction cosines are

$\langle \frac{A}{\pm\sqrt{A^2+B^2+C^2}}, \frac{B}{\pm\sqrt{A^2+B^2+C^2}}, \frac{C}{\pm\sqrt{A^2+B^2+C^2}} \rangle$

$\therefore d = projection\ of\ line\ segment\ \overline{PQ}\ on\ \overrightarrow{QN}\ .\ \overrightarrow{QN}$

$= \frac{A}{\pm\sqrt{A^2+B^2+C^2}}\,(x_0 - \alpha) + \frac{B}{\pm\sqrt{A^2+B^2+C^2}}\,(y_0 - \beta) + \frac{C}{\pm\sqrt{A^2+B^2+C^2}}\,(z_0 - \gamma)$

$= \frac{A(x_0 - \alpha)+B(y_0-\beta)+C(z_0-\gamma)}{\pm\sqrt{A^2+B^2+C^2}}$

$= \frac{Ax_0+By_0+Cz_0-(A\alpha+B\beta+C\gamma)}{\pm\sqrt{A^2+B^2+C^2}}$

Now (α, β, γ) *lies on the given plane* $(A\alpha + B\beta + C\gamma + D = 0$ hence $(A\alpha + B\beta + C\gamma = -D$

Thus

d = $\frac{Ax_0+By_0+Cz_0+D)}{\pm\sqrt{A^2+B^2+C^2}}$

The sign of the denominator chosen accordingly so as to make the whole quantity positive. In particular the distance of the plane from the origin is given by

$\frac{D}{\pm\sqrt{A^2+B^2+C^2}}$

Example

Find the distance d from the point P(7,5,1) to the plane 9x+3y-6z-2 = 0

Solution :

Let R be any point of the plane. The scalar projection of vector $\overline{RP}$ on a vector perpendicular to the plane gives the required distance. The scalar projection is obtained by taking the dot project of $\overline{RP}$ and a unit vector normal to the plane. The point (1,0,0) is in the plane and using this point for R, we have $\overline{RP}$ = 6i+5j+k

N = $\pm\ \frac{2i+3j-6k}{7}$ is a unit vector normal to the plane. Hence

N . $\overline{RP}$ $= \pm \frac{12+15-6}{7} = \ \pm \frac{21}{7}$ we choose the ambiguous sign + in order to have a positive result. Thus we get d = 3.

Dihedral angle(Angle Between two planes)

Given two intersecting planes described by

$$\Pi_1 : a_1 x + b_1 y + c_1 z + d_1 = 0 \quad \text{and}$$

$$\Pi_2 : a_2 x + b_2 y + c_2 z + d_2 = 0,$$

the dihedral angle between them is defined to be the angle α between their normal directions:

$$\cos\alpha = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1||\hat{n}_2|} = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Example : Find the angle θ between the plane 4x-y+8z+7 =0 and x + 2y-2z+5 = 0

Solution :

The angle between two plane is equal to the angle between their normals. The vectors

$N_1 = \frac{4i-j+8k}{9} \quad N_2 = \frac{i+2j-2k}{3}$

Are unit vectors normal to the given planes. The dot product yield

$\cos\theta = N_1 . N_2 = -\frac{14}{27}$ or $\theta = 121^o$

Equation Of plane passing through three given point

Let (x_1, y_1, z_1) , (x_2, y_2, z_3) and (x_3, y_3, z_3) be three given points and the required plane be Ax+By+Cz+D =0……………….. (1)

Since it passes through (x_1, y_1, z_1) we have

Ax_1 + By_1+ Cz_1 = 0 ………(2)

Subtracting eq (2) from (1) we have

A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 ………….(3)

Since this plane also passes through (x_2, y_2, z_3) and (x_3, y_3, z_3) we have

$A(x_2 - x_1) + B\,(y_2 - y_1) + \,C(z_2 - \,z_1) = 0$

And

$A(x_3 - x_1) + B\,(y_3 - y_1) + \,C(z_3 - \,z_1) = 0$ ………… (5)

Eliminating A,B, C from eq (3),(4) and (5) we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

Which is the equation of the plane

Corollary 1:

If the plane makes the intercepts a,b,c on the co-ordinate axes $\overleftrightarrow{OX}, \overleftrightarrow{OY}, \overleftrightarrow{OZ}$ respectively then the plane passes through the point (a,0,0) , (0.b,0) and (0,0,c). Hence the equation (6) gives

$$\begin{vmatrix} x - a & y - 0 & z - 0 \\ 0 - a & b - 0 & 0 - 0 \\ 0 - a & 0 - 0 & c - 0 \end{vmatrix} = \begin{vmatrix} x - a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \text{bc(x-a) +yac+zab = o}$$

Dividing both side by abc we get $\frac{(x-a)}{a} + \frac{y}{b} + \frac{z}{c} = 0$

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Example :

Find the equation of the plane determined by the point $P_1(2,3,7), P_2(2,3,7), P_3(2,3,7)$

Solution :

A vector which is perpendicular to two sides of tringle $P_1P_2P_3$ is normal to the plane of the tringle. To find the vector we write

$\overline{P_1P_2} = 3i + 2j - 5k \quad P_1P_3 = i + j - k$ N = Ai+Bj+Ck

The coeff A,B,C are to be found so that N is perpendicular to each of the vector Thus

N. P_1P_2 = 3A+2B-5C = 0

N. P_1P_3 = A+B-C = 0

These equation gives A = 3C and B = -2C. Choosing C =1 we have N = 3i-2j+k. Hence tha plane 3x-2y+z+D = 0 is normal to N and passing through the points if D =-7

Hence the equation is 3x-2y+z-7 = 0.

Alternate :

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = \begin{vmatrix} x-2 & y-3 & z-7 \\ 5-2 & 5-3 & 2-7 \\ 3-2 & 4-3 & 6-7 \end{vmatrix} == \begin{vmatrix} x-2 & y-3 & z-7 \\ 3 & 2 & -5 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

=(x-2)(-2+5)-(y-3)(-3+5)+(z-7)(3-2) =0

= 3x-6-2y+6+z-7= 0

Or 3x -2y+z -7 = 0

Exercise

1. Write the equation of the plane perpendicular to N = 2i-3j+5k and passing through the point (2,1,3) Ans 2x-3y+5z-6 = 0

2. Parallel to the plane 3x-2y-4z = 5 and passing through (2,1,-3) Ans 3x-2y-4z-16 = 0

3. Passing through the (3,-2,-1)(-2,4,1)(5,2,3) Ans 2x+3y-4z-4 = 0

4. Find the perpendicular distance from 2x-y+2z+3 = 0 (1,0,3) Ans :$\frac{11}{3}$

5. Find the perpendicular distance from 4x-2y+z-2 =0 (-1,2,1) Ans : $\frac{9}{\sqrt{21}}$

6. Find the cosine of the acute angle between each pair of plane 2x+2y+z-5 = 0 , 3x-2y+6z+5 = 0

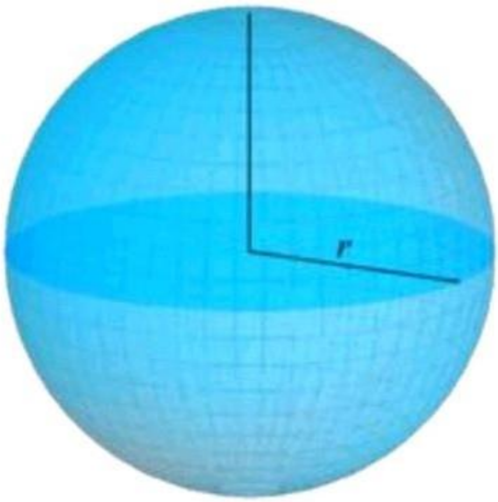
Ans : $\frac{8}{21}$

7. Find the cosine of the acute angle between each pair of plane 4x-8y+z-3 = 0, 2x+4y-4z+3 = 0

Ans : $\frac{14}{27}$

Sphere

A **sphere** (from Greek σφαῖρα — *sphaira*, "globe, ball"[1]) is a perfectly round geometrical and circular object in three-dimensional space that resembles the shape of a completely round ball. Like a circle, which, in geometric contexts, is in two dimensions, a sphere is defined mathematically as the set of points that are all the same distance *r* from a given point in three-dimensional space. This distance *r* is the radius of the sphere, and the given point is the center of the sphere. The maximum straight distance through the sphere passes through the center and is thus twice the radius; it is the diameter.



Equation Of a Sphere

Fig

Let the centre of the sphere be the point (a,b,c) and the radius of be r(x,y,z) be any point on the sphere. By distance formula

$CP^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

Or, $(x-a)^2+(y-b)^2+(z-c)^2 = r^2$ ..... (1)

Is the required equation.

In particular , if instead of any point (a,b,c) the centre of the sphere is the origin (0,0,0) and redius r then from (1), we obtain by putting a=b=c=0 , the equation of the sphere as

$x^2+y^2+z^2 = r^2$ .....(2)

Since the equation (1) can be further be written as

$x^2+y^2+z^2-2ax-2by-2cz+a^2+b^2+c^2=r^2$

Or, $x^2+y^2+z^2-2ax-2by-2cz+d = 0$

Where d = $a^2+b^2+c^2-r^2$

We conclude that

(i) The equation of a sphere is of second order in x,y,z
(ii) The coefficient of $x^2, y^2 and\ z^2$ are equal
(iii) There is no term containing xy,yz, or zx

General Equation :

Consider the general equation of second degree in x,y,z

$x^2+y^2+z^2+2ux+2vy+2wz+d = 0$ .....(3)

This can be written as

$(x+u)^2+(y+v)^2+(z+w)^2 = u^2+v^2+w^2-d$

Which on comparison with eq (1) implies that the equation represent a sphere with centre(-u,-v,-w) and radius = $\sqrt{u^2+v^2+w^2-d}$

In general the equation

$x^2+y^2+z^2+Dx+Ey+Fz+d = 0$ ....(4)

Represent a sphere with centre $(-\frac{D}{2}, -\frac{E}{2}, -\frac{F}{2})$ and radius = $\sqrt{\frac{D^2}{4}+\frac{E^2}{4}+\frac{F^2}{4}-G}$

Since equation(3) contains four arbitrary constants, we need four non-coplanar points to determine a sphere uniquely.

Sphere through four non-coplanar points

Let A$(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3) and\ D(x_4, y_4, z_4,)$ be four given non-coplannar points and the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Since the sphere passes through four given points the co-ordinate of the given points satisfy the equation of the sphere and hence , we have

$$x^2 + y^2 + z^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$$x^2 + y^2 + z^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0$$

$$x^2 + y^2 + z^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0$$

$$x^2 + y^2 + z^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0$$

Solving these four simultaneous equation for u,v,w and d we obtain the equation of the required sphere.

Example –

Find the equation of the sphere through points (0,0,0), (0,1,-1) and (1,2,3). Locate its centre and find the radius?

Solution :

Let the required equstion of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

It passes through (0,0,0), (0,1,-1),(-1,2,0) and (1,2,3)

d = 0 ……(1)

1+1+2v-2w+d = 0

or, v – w = -1……(2)

1+4-2u+4v+d = 0

or, 2u – 4v = 5 ……(3)

1+4+9+2u+4v+6w+d = 0

or, u+2v+3w = -7 ……(4)

Solving the above equation we get u = $-\frac{15}{14}$, v =$-\frac{25}{14}$, w = $-\frac{11}{14}$

Hence by substituting the above equation the required eq

$x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0$

Its centre at ($-\frac{15}{14}$, $-\frac{25}{14}$, $-\frac{11}{14}$)

And the radius = $\sqrt{\left(-\frac{15}{14}\right)^2 + \left(-\frac{25}{14}\right)^2 + \left(-\frac{11}{14}\right)^2} = \frac{\sqrt{971}}{14}$

Equation of a sphere on a given diameter

Fig

Let A$(x_1, y_1, z_1), B(x_2, y_2, z_2)$ be the end points of a diameter of the sphere. If we consider P(x,y,z) any point on the sphere,then m<APB = 90^0 ie. $\overline{AP}$ is perpendicular to . $\overline{PB}$. Since the direction ratio of $\overleftrightarrow{AP}$ *and* $\overline{PB}$ are <x-x_1, y-y_1, $z - z_1$ > and <x-x_2, y-y_2, $z - z_2$ > respectively by condition of perpenticularity we have

(x-x_1) (x-x_2) +(y-y_1)($y - y_2$)+(z-z_1) (z-z_2) = 0

Example :

Deduce the equation of the sphere described on line joining the points (2,-1,4) and (-2,2,-2) as diameter.

Solution :

Let P(x,y,z) be any point on the sphere having A$(x_1, y_1, z_1), B(x_2, y_2, z_2)$ as ends of diameter. Then AP and BP are at right angle.

Now the direction ratio are (x-x_1), (y-y_1) , (z-z_1)

And those of BP are (x-x_2), ($y - y_2$), (z-z_2)

Hence (x-x_1) (x-x_2) +(y-y_1)($y - y_2$)+(z-z_1) (z-z_2) = 0 is the required equation.

The equation of the required sphere is

(x-2)(x+2) +(y+1)(y+2)+(z-4)(z+2) = 0

Or $x^2 + y^2 + z^2$-y-2z-14 = 0

Problems

1. Find the equation of the sphere through the point (2,0,1), (1,-5,-1), (0,-2,3) and (4,-1,2). Also find its centre and radius
   (Ans : $x^2 + y^2 + z^2$-4x+6y-2z+5 = 0; (2,-3,-1); 3)
2. Obtain the equation of the sphere passing through the origin and the points (a,0,0), (0,b,0) and (0,0,c).
   (Ans : $x^2 + y^2 + z^2 - ax - by - cz = 0$)
3. Find the equation of the sphere whose diameter is the line joining the origin to the point (2,-2,4)
   (Ans $x^2 + y^2 + z^2$-2x+2y-4z = 0)