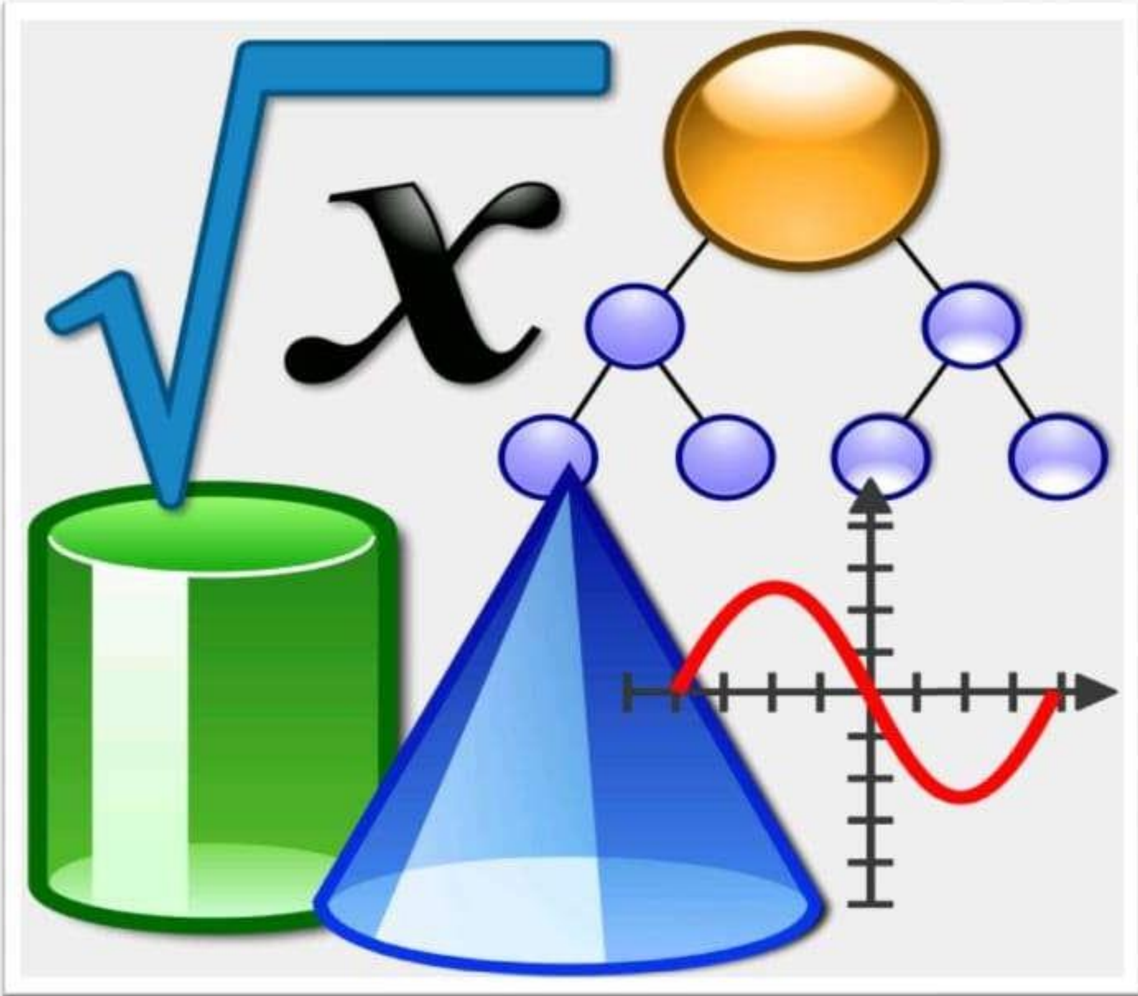
DEPARTMENT OF BASIC SCIENCE

ENGINEERING MATHEMATICS-III



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CONTENTS

CHAPTER - 1

TRIGONOMETRY

COMPOUND ANGLES

INTRODUCTION :

The word Trigonometry is derived from Greek words ''Trigonos'' and metrons means measurement of angles in a triangle. This subject was originally devecpaed to solve geometric problems involving trigangles. The Hindu mathematicians Aryabhatta, Varahmira, Bramhaguptu and Bhaskar have lot of contaribution to trigonometry . Besides Hindu mathematicians ancient-Greek and Arwric mathematicians also contributed a lot to this subject. Trigonometry is used in many are as such as science of seismology, designing electrical circuits, analysing musical tones and studying the occurance of sun spots.

Trigonometric Functions :

Let θ be the meausre of any angle measured in radians in counter clockmise sense as show in Fig (1).

Let P(x, y) be any point an the terminal side of angle θ. The distance of P from O is OP = r = $\sqrt{x^2+y^2}$. the functions defined by $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$ ...(1) are called sine, cosine and tangent functions respectirely. These are called trigonometric functions. It followrs from (1) that $\sin^2\theta + \cos^2\theta = 1$. Other trigonomatric functions such as cosecant, secant and cotangent functions are defined as $\text{cosec}\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$.

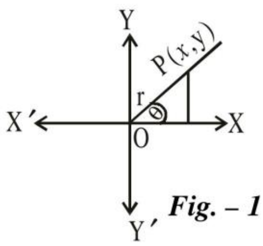
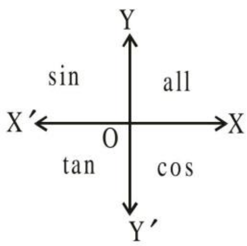


Fig. – 1

SIGN OF T-RATIOS :

The student may remember the signs of t-ratios in different quadrant with the help of the diagram



The sign of paricular t-ratio in any quadrant can be remembered by the word ''all-sin-tan-cos'' or ''add sugar to coffee''. What ever is written in a particular quadrant along with its reciprocal is +ve and the rest are negetive.

Table giving the values of trigonometrical Ratios of angles 0º, 30°, 45º, 60º & 90º

θ	0º	30º	45º	60º	90º
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

RELATED ANGLES :

Definitions : Two angles are said to be complementary angles if their sum is 90° and each angle is said to be the complement of the other.

Two angles are said to be supplementary if their sum is 180° and each angles is said to be the supplement of the other.

To Find the T-Ratios of angle (–θ) in terms of θ :

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle θ in the anticlockwise sense which we take as positive sense. ***(Fig. 2)***

Let OP' be the position of the radius vector after tracing (θ) in the clockwise sense, which we take as negative sense. So $\angle$P'OX will be taken as –θ. Join PP'. Let it meet OX at M.

Now Δ OPM $\equiv$ Δ P'OM, $\angle$P'OM $= -\theta$

OP' = OP, P'M = – PM

Now $\sin(-\theta) = \frac{P'M}{OP'} = \frac{-PM}{OP} = -\sin\theta$

$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta$

$\tan(-\theta) = \frac{P'M}{OM} = \frac{-PM}{OM} = -\tan\theta$

$\operatorname{cosec}(-\theta) = \frac{OP'}{P'M} = \frac{OP}{-PM} = -\operatorname{cosec}\theta$

$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta$

$\cot(-\theta) = \frac{OM}{P'M} = \frac{OM}{-PM} = -\cot\theta$

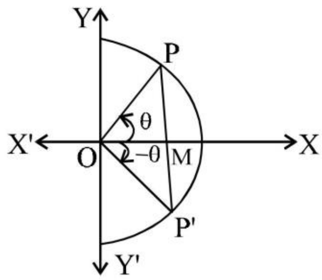


Fig. – 2

To find the T-Ratios of angle (90° – θ) in terms of θ.

Let OPM be a right angled triangle with $\angle$POM = 90°, $\angle$OMP = θ,

$\angle$OPM = 90°– θ. ***(Fig. 3)***

$\therefore$ $\sin(90° - \theta) = \frac{OM}{PM} = \cos\theta \Rightarrow \operatorname{cosec}(90° - \theta) = \sec\theta$

$\cos(90° - \theta) = \frac{OP}{PM} = \sin\theta \Rightarrow \sec(90° - \theta) = \operatorname{cosec}\theta$

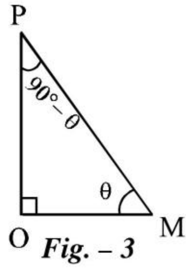


Fig. – 3

$$\tan(90° - \theta) = \frac{OM}{OP} = \cot\theta \quad \Rightarrow \quad \cot(90° - \theta) = \tan\theta$$

To find the T-Ratios of angle (90° + θ) in terms of θ.

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$. Draw PM and P'M' perpendiculars to the X-axis***(Fig. 4)***

Now $\Delta POM \cong \Delta P'OM'$

$\therefore$ P'M' = OM and OM' = –PM

Now $\sin(90° + \theta) = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos\theta$

$\cos(90° + \theta) = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin\theta$

$\tan(90° + \theta) = \frac{P'M'}{OM'} = \frac{OM}{-PM} = -\cot\theta$

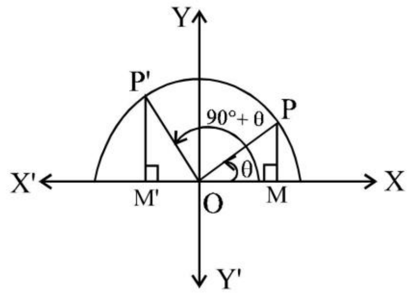


Fig. – 4

Similarly cosec (90° + θ) = sec θ

sec (90° + θ) = – cosec θ

and cot (90° + θ) = – tan θ

To Find the T-Ratios of angle (180° – θ) in terms of θ.

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle XOP = θ To obtain the angle 180° – θ let the radius vector start from OX and after revolving through 180° come to the position OX'. Let it revolve back through an angle θ in the clockwise direction and come to the position OP' so that the angle X'OP' is equal in magnitude but opposite in sign to the angle XOP. The angle XOP' is 180° – θ. ***(Fig.5)***

Draw P'M' and PM perpendicular to X'OX.

Now $\Delta POM \equiv \Delta P'OM'$.

$\therefore$ OM' = –OM and P'M' = PM

Now $\sin(180° - \theta) = \frac{P'M'}{OP'} = \frac{PM}{OP} = \sin\theta$

$\cos(180° - \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$

$\tan(180° - \theta) = \frac{P'M'}{OM'} = \frac{PM}{-OM} = -\tan\theta$

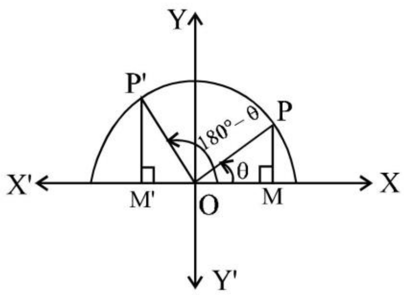


Fig. – 5

Similarly cosec (180° – θ) = cosec θ

sec (180° – θ) = –sec θ

and cot (180° – θ) = – cot θ

To Find the T-Ratios of angle (180° + θ) in terms of θ.

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$.***(Fig. 6)***

Now $\Delta POM \equiv \Delta P'OM'$.

$\therefore$ OM' = –OM and P'M' = –PM

Now $\sin(180° + \theta) = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin\theta$

$\cos(180° + \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$

$\tan(180° + \theta) = \frac{P'M'}{OM'} = \frac{-PM}{-OM} = \tan\theta$

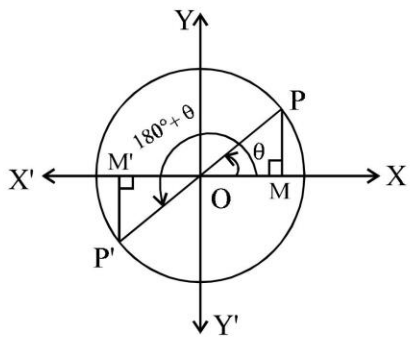


Fig. – 6

Similarly cosec $(180° + \theta)$ = cosec θ

sec $(180° + \theta) = -\sec\theta$

and cot $(180° + \theta) = \cot\theta$.

To Find the T-Ratios of angles (270° $\pm$ θ) in terms of θ.

The trigonometrical ratios of $270° - \theta$ and $270° + \theta$ in terms of those of θ, can be deduced from the above articles. For example

$\sin(270° - \theta) = \sin[180° + (90° - \theta)]$

$= -\sin(90° - \theta) = -\cos\theta$

$\cos(270° - \theta) = \cos[180° + (90° - \theta)]$

$= -\cos(90° - \theta) = -\sin\theta$

Similarly $\sin(270° + \theta) = \sin[180° + (90° + \theta)]$

$= -\sin(90° + \theta) = -\cos\theta$

$\cos(270° + \theta) = \cos[180° + (90° + \theta)]$

$= -\cos(90° + \theta) = -(-\sin\theta) = \sin\theta$

To Find the T-Ratios of angles (360° $\pm$ θ) in terms of θ.

We have seen that if n is any integer, the angle n. 360° $\pm\,\theta$ is represented by the same position of the radius vector as the angle $\pm\,\theta$. Hence the trigonometrical ratios of 360° $\pm\theta$ are the same as those of $\pm\theta$.

Thus $\sin(n.\,360° + \theta) = \sin\theta$

$\cos(n.\,360° + \theta) = \cos\theta$

$\sin(n.\,360° - \theta) = \sin(-\theta) = -\sin\theta$

and $\cos(n.\,360° - \theta) = \cos(-\theta) = \cos\theta$.

Examples :

$\cos(-720° - \theta) = \cos(-2 \times 360° - \theta) = \cos(-\theta) = \cos\theta$

and $\tan(1440° + \theta) = \tan(4 \times 360° + \theta) = \tan\theta$

In general when is any integer, $n \in Z$

(1) $\sin(n\pi + \theta) = (-1)^n \sin\theta$

(2) $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(3) $\tan(n\pi + \theta) = \tan\theta$ when n is odd integer

(4) $\sin\left(\frac{n\pi}{z} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$

(5) $\cos\left(\frac{n\pi}{z} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$

(6) $\tan\left(\frac{n\pi}{z} + \theta\right) = \cot\theta$

EVEN FUNCTION :

A function $f(x)$ is said to be an even function of x, if $f(x)$ satisfies the relation $f(-x) = f(x)$.

Ex. cosx, secx, and all even powers of x i.e, x^2, x^4, x^6...... are even function.

ODD FUNCTION :

A function $f(x)$ is said to be an odd function of x, if $f(x)$ satisfies the relation $f(-x) = -f(x)$.

Ex. sinx, cosec x, tan x, cot x and all odd powers of x i.e, x^3, x^5, x^7...... are odd function.

Example : *Find the values of $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{-12}{13}$ and θ lies in the third quadrant.*

Solution : We have $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$$

In third quadrant $\sin\theta$ is negetive, therefore

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(\frac{-12}{13}\right)^2} = \frac{-5}{3}$$

Now $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-5}{13} \times \frac{13}{-12} = \frac{5}{12}$

Example : Find the values of

(i) tan (–900º) **(ii) sin 1230º**

Solution : (i) tan (–900º) = –tan 900º = – tan (10 × 90º + 0º) = – tan0º = 0

(ii) sin (1230º) = sin (6 × 180º + 150º) = sin 150º = sin (180º – 30º) = sin 30º = $\frac{1}{2}$

Example : Show that

$$\frac{\cos(90^\circ+\theta).\sec(-\theta).\tan(180^\circ-\theta)}{\sec(360^\circ-\theta).\sin(180^\circ+\theta)\ .\cot(90^\circ-\theta)} = -1 = \frac{-\sin\theta\times\sec\theta\times-\tan\theta}{\sec\theta\times-\sin\theta\times\tan\theta} = -1$$

Solution : $\frac{\cos(90^\circ+\theta).\sec(-\theta).\tan(180^\circ-\theta)}{\sec(360^\circ-\theta).\sin(180^\circ+\theta)\ .\cot(90^\circ-\theta)} = \frac{-\sin\theta\times\sec\theta\times-\tan\theta}{\sec\theta\times-\sin\theta\times\tan\theta} = -1$

ASSIGNMENT

1. Find the value of cos1º.cos2º..... cos 100º

2. Evaluale : $\tan\frac{\pi}{20}\cdot\tan\frac{3\pi}{20}\cdot\tan\frac{5\pi}{20}\cdot\tan\frac{7\pi}{20}\cdot\tan\frac{9\pi}{20}\cdot$

❐ ☸ ❐

COMPOUND, MULTIPLE AND SUB-MULTIPLE ANGLES

When an angle formed as the algebric sum of two or more angles is called a compound angles. Thus A + B and A + B + c are compound angles.

Addition Formulae

When an angle formed as the algebraical sum of two or more angles, it is called a compound angles. Thus A + B and A + B + C are compound angles.

Addition Formula :

(i) $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

(ii) $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

(iii) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

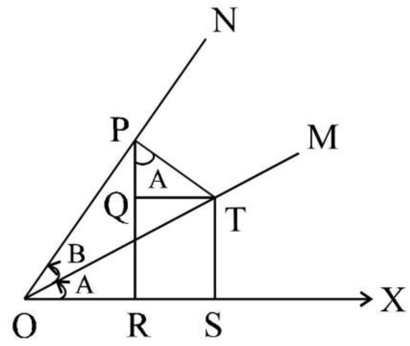


Fig. – 7

Proof : Let the revolving line OM starting from the line OX make an angle XOM = A and then further move to make.

$\angle MON = B$, so that $\angle XON = A + B$ ***(Fig. 7)***

Let 'P' be any point on the line ON.

Draw $PR \perp OX$, $PT \perp OM$, $TQ \perp PR$ and $TS \perp OX$

Then $\angle QPT = 90° - \angle PTQ = \angle QTO = \angle XOM = A$

$\therefore$ We have from Δ OPR

(i) $\sin(A + B) = \dfrac{RP}{OP} = \dfrac{QR + PQ}{OP} = \dfrac{TS + PQ}{OP}$ $\quad$ ($\because$ QR = TS)

$= \dfrac{TS}{OP} + \dfrac{PQ}{OP} = \dfrac{TS}{OT} \cdot \dfrac{OT}{OP} + \dfrac{PQ}{PT} \cdot \dfrac{PT}{OP}$

$= \sin A \cdot \cos B + \cos A \cdot \sin B$

(ii) $\cos(A + B) = \dfrac{OR}{OP} = \dfrac{OS - RS}{OP} = \dfrac{OS}{OP} - \dfrac{RS}{OP}$

$= \dfrac{OS}{OP} - \dfrac{QT}{OP}$ $\quad$ [$\because$ RS = QT]

$= \dfrac{OS}{OT} \cdot \dfrac{OT}{OP} - \dfrac{QT}{PT} \cdot \dfrac{PT}{OP}$

$= \cos A \cdot \cos B - \sin A \cdot \sin B$

(iii) $\tan(A + B) = \dfrac{\sin(A + B)}{\cos(A + B)}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B)}$

(dividing numerator and denominator by cos A cos B)

$= \dfrac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

(iv) $\cot(A+B) = \dfrac{\cos(A+B)}{\sin(A+B)}$

$= \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

[dividing of numerator and denominator by sin A sin B]

$= \dfrac{\dfrac{\cos A \cos B}{\sin A \sin B} - 1}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}}$

$\cot(A+B) = \dfrac{\cot A . \cot B - 1}{\cot B + \cot A}$

Cor : In the above formulae, replacing A by $\dfrac{\pi}{2}$ and B by x

We have

(i) $\sin\left(\dfrac{\pi}{2} + x\right) = \sin\dfrac{\pi}{2} . \cos x + \cos\dfrac{\pi}{2} . \sin x$

$= 1 . \cos x + 0 . \sin x = \cos x$

(ii) $\cos\left(\dfrac{\pi}{2} + x\right) = \cos\dfrac{\pi}{2} . \cos x - \sin\dfrac{\pi}{2} . \sin x$

$= 0 \times \cos x - 1 \times \sin x = -\sin x$

(iii) $\tan\left(\dfrac{\pi}{2} + x\right) = \dfrac{\sin\left(\dfrac{\pi}{2} + x\right)}{\cos\left(\dfrac{\pi}{2} + x\right)} = \dfrac{\cos x}{-\sin x} = -\cot x$

(b) Difference Formulae :

(i) $\sin(A - B) = \sin A . \cos B - \cos A . \sin B$

(ii) $\cos(A - B) = \cos A . \cos B + \sin A . \sin B$

(iii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A . \tan B}$

Proof : Let the reveolving line OM make an angle A with OX and then resolve back to make $\angle MON = B$ so that $\angle XON = A - B$. ***(Fig. 8)***

Let 'P' be any point on ON. Draw $PR \perp OX$, $PT \perp OM$, $TS \perp OX$, $TQ \perp RP$ produced to Q.

Then $\angle TPQ = 90° - \angle PTQ = \angle QTM = A$

Now from Δ OPR, we have

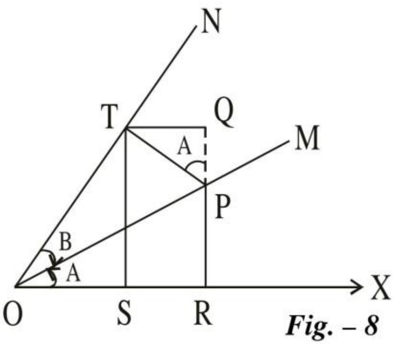


Fig. – 8

(i) $\sin(A - B) = \dfrac{PR}{OP} = \dfrac{QR - QP}{OP} = \dfrac{TS - QP}{OP}$

$$= \frac{TS}{OP} - \frac{QP}{OP}$$

$$= \frac{TS}{OT} \cdot \frac{OT}{OP} - \frac{OP}{PT} \cdot \frac{PT}{OP}$$

= sin A . cos B – cos A . sin B

(ii) $\cos(A - B) = \frac{OR}{OP} = \frac{OS + SR}{OP} = \frac{OS + TQ}{OP} = \frac{OS}{OP} + \frac{TQ}{QP}$

$$= \frac{OS}{OT} \cdot \frac{OT}{OP} + \frac{TQ}{PT} \cdot \frac{PT}{OP}$$

= cos A . cos B + sin A . sin B

(iii) $\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A . \cos B - \cos A . \sin B}{\cos A \cos B + \sin A \sin B}$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Dividing the numerator and the denominator by cos A. cos B.

(iv) $\cot(A - B) = \frac{\cos(A - B)}{\sin(A - B)}$

$$= \frac{\cos A . \cos B + \sin A . \sin B}{\sin A . \cos B - \cos A . \sin B}$$

$$= \frac{\cot A . \cot B + 1}{\cot B - \cot A}$$

dividing the numerator and denominator by sin A. sin B

We can also deduce substraction formulae from addition formulae in the following manner.

sin(A – B) = sin[A + (–B)]
= sin A . cos (–B) + cos A . sin (–B)
= sin A . cos B + cos A . sin B
cos(A – B) = cos[A + (–B)]
= cos A . cos (–B) – sin A . sin (–B)
= cos A . cos B + sin A . sin B

$$\tan(A - B) = \tan[A + (-B)] = \frac{\tan A + \tan(-B)}{1 - \tan A . \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$$

Example – 1 : **Find the value of tan 75° and hence prove that tan 75° + cot 75° = 4**

Solution: $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1 \times 1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$\therefore \quad \tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\therefore \quad \cot 75° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} \quad \left(\text{since } \cot\theta = \dfrac{1}{\tan\theta}\right)$

$$\tan 75° + \cot 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{3-1} \qquad [\text{since } (a+b)(a-b) = a^2 - b^2]$$

$\therefore \quad \tan 75° + \cot 75° = 4$

Example – 2 : **If** $\sin A = \dfrac{1}{\sqrt{10}}$ **and** $\sin B = \dfrac{1}{\sqrt{5}}$ **show that** $A + B = \dfrac{\pi}{4}$

Solution: $\sin A = \dfrac{1}{\sqrt{10}}$

$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{1}{10}} = \sqrt{\frac{10-1}{10}} = \sqrt{\frac{9}{10}}$$

$$\therefore \cos A = \frac{3}{\sqrt{10}}$$

$$\sin B = \frac{1}{\sqrt{5}}, \cos B = \sqrt{1-\sin^2 B}$$

$$= \sqrt{1-\frac{1}{5}} = \sqrt{\frac{5-1}{5}} = \sqrt{\frac{4}{5}}$$

$$\therefore \cos B = \frac{2}{\sqrt{5}}$$

$\text{Sin}(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{1}{\sqrt{10}} \times \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{50}} + \frac{3}{\sqrt{50}}$$

$$= \frac{2+3}{\sqrt{50}} = \frac{2+3}{5\sqrt{2}}$$

$$\therefore \sin(A+B) = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\sin(A + B) = \sin 45°$

$$\therefore A + B = 45° = \frac{\pi}{4} \left[\text{since } 45° = \frac{180°}{4}\right]$$

Transformation of Sums or Difference in to Products

(a) We have that

$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B \quad ....(1)$

$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B \quad ....(2)$

$\cos(A + B) - \cos(A - B) = 2 \cos A \cos B \quad ....(3)$

$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B \quad ....(4)$

Let $A + B = C$ and $A - B = D$

Then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$

Putting the value in formula (1), (2), (3) and (4) we get

$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$ .... (i)

$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$ .... (ii)

$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$ .... (iii)

$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$ .... (iv)

for practice it is more convenient to quote the formulae verbally as follows :

Sum of two sines = 2 sin (half sum) cos (half difference)

Difference of two sines = 2 cos (half sum) sin (half difference)

Sum of two cosines = 2 cos (half sum) cos (half difference)

Difference of two cosines = 2 sin (half sum) sin (half difference reversed)

[The student should carefully notice that the second factor of the right hand member of IV is sin $\frac{D-C}{2}$, not sin $\frac{C-D}{2}$]

(b) To find the Trigonometrical ratios of Angle 2A in terms of those of A : sin 2A, cos 2A.

Since sin (A + B) = sin A cos B + cos A sin B

putting B = A

sin (A + A) = sin A cos A + cos A sin A

$\Rightarrow$ sin 2A = 2 sin A cos A ..... (i)

cos (A + B) = cos A cos B – sin A sin B

$\Rightarrow$ cos (A + A) = cos A cos A – sin A sin A

$\Rightarrow$ cos 2A = $\cos^2 A - \sin^2 A$ ..... (ii)

Also cos 2A = $1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$ ..... (iii)

So $2\sin^2 A = 1 - \cos 2A$ ..... (iv)

Also cos 2A = $\cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$ ..... (v)

or $2\cos^2 A = 1 + \cos 2A$ ..... (vi)

(c) Formula for tan 2A

since $\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan 2A = \tan (A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$

$= \frac{2 \tan A}{1 - \tan^2 A}$

Note this formula is not defined when $\tan^2 A = 1$ i.e, $\tan A = \pm 1$

(d) To express sin 2A and cos 2A in terms of tan A

sin 2A = 2 sin A cos A

$$= 2 \frac{\frac{\sin A}{\cos A}}{\frac{1}{\cos^2 A}} = \frac{2 \tan A}{\sec^2 A} = \frac{2 \tan A}{1 - \tan^2 A}$$

Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(dividing numerator and denominator by $\cos^2 A$)

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(e) To find the Trigonometrical formulae of 3A

$\sin 3A = \sin (2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2 \sin A \cos A . \cos A + (1 - 2 \sin^2 A) \sin A$

$= 2 \sin A(1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A$

$= 3 \sin A - 4 \sin^3 A$

Again, $\cos 3A = \cos (2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A. \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \cos A (1 - \cos^2 A)$

$= 4 \cos^3 A - 3 \cos A$

Also $\tan 3A = \tan (2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A}.\tan A}$$

$$= \frac{2 \tan A + \tan A(1 - \tan^2 A)}{1 - \tan^2 A - 2\tan^2 A}$$

$= \dfrac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$, provided $3 \tan^2 A \neq 1$ i.e, $\tan A \neq \pm \dfrac{1}{\sqrt{3}}$

(f) Submultiple Angles :

To express trigonometric ratios of A in terms of ratios of A/2

$\sin 2\theta = 2 \sin \theta \cos \theta$ (true for all value of θ)

Let $2\theta = A$ i.e. $\theta = \dfrac{A}{2}$

$\sin A = 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$ .... (i)

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

or $\cos A = \cos^2 \dfrac{A}{2} - \sin^2 \dfrac{A}{2}$ ..... (ii)

$\cos A = 2 \cos^2 \dfrac{A}{2} - 1 = 1 - 2 \sin^2 \dfrac{A}{2}$ ..... (iii)

Also, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \qquad \text{..... (iv)}$$

[Where $A \neq n\pi + \dfrac{\pi}{2}$, $(n \in I)$ and $A \neq (2n+1)\pi$]

Again, $\sin A = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1} = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

[dividing numerator and denomenator by $\cos^2\dfrac{A}{2}$)

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

[where $A \neq (2n+1)\pi$, $n \in I$]

Similarly, $\cos A = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{1} = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

Now dividing numerator and denominator by $\cos^2\dfrac{A}{2}$

$\Rightarrow \quad \cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$ [where $A \neq (2n+1)\pi$, $n \in I$].

Example –1 : Find the values of

(i) $\mathbf{\cos 22\frac{1}{2}^{o}}$ **(ii)** $\mathbf{\sin 15^{o}}$

Solution : **(i)** We have c $\cos\dfrac{A}{2} = \sqrt{\dfrac{1+\cos A}{2}}$, putting A = 45º

$$\cos 22\frac{1}{2}^{o} = \sqrt{\frac{1+\cos 45^{o}}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(ii) sin 15º = sin (45º – 30º)

= sin 45º. cos30º – cos45º. sin 30º

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example – 2: Prove that sinA.sin (60º – A).sin(60º + A) = $\frac{1}{4}$ sin3A

Solution : sinA.sin (60º – A) sin (60º + A)

$= \sin A.\ (\sin^2 60^\circ - \sin^2 A)$ $[\because \sin (A + B). \sin (A - B) = \sin^2 A - \sin^2 B]$

$$= \sin A\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A\right] = \sin\left[\frac{3}{4} - \sin^2 A\right] = \frac{1}{4} \quad [3\sin A - 4\sin^3 A]$$

$$= \frac{1}{4}\sin 3A$$

Example – 3: Prove that sin20º.sin40º.sin60º.sin80º = $\frac{3}{16}$

Solution : sin 60º. sin 20º.sin40º. sin 80º

$$= \frac{\sqrt{3}}{2}[\sin A.\ \sin (60^\circ - A).\ \sin (60^\circ + A)] \text{ where } A = 20^\circ$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{4}\cdot\sin 3A = \frac{\sqrt{3}}{8}\cdot\sin 60 = \frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{2} = \frac{3}{16}$$

Example – 4: If A + B + C = π and cos A = cos B . cos C show that tan B + tan C = tan A

Solution : L.H.S. = tan B + tan C

$$= \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin B . \cos C + \cos B . \sin C}{\cos B . \cos C}$$

$$= \frac{\sin (B + C)}{\cos B . \cos C} = \frac{\sin (\pi - A)}{\cos B . \cos C} = \frac{\sin A}{\cos A} = \tan A = \text{R.H.S. (Proved)}$$

Examples – 5: Prove the followings

(a) $\mathbf{\cot 7\frac{1^\circ}{2} = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2}$

(b) $\mathbf{\tan 37\frac{1^\circ}{2} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2}$

Solution : **(a)** We know $\cot\frac{\theta}{2} = \frac{1 + \cos\theta}{\sin\theta}$ (Choosing θ = 15)

$$= \cot 7\frac{1}{2}{}^\circ = \frac{1 + \cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1 + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2 + \sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{(2\sqrt{2} + \sqrt{3} + 1)\,(\sqrt{3} + 1)}{(\sqrt{3} - 1)\,(\sqrt{3} + 1)} = \frac{2\sqrt{6} + 2\sqrt{2} + \sqrt{3} + \sqrt{3} + 1 + 3}{3 - 1}$$

$$= \frac{2\sqrt{6} + 2\sqrt{3} + 2\sqrt{2} + 4}{2} = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$$

(b) We know $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ (Choosing θ = 75º)

$$\tan 37\frac{1^\circ}{2} = \frac{1-\cos 75^\circ}{\sin 75^\circ} = \frac{1-\cos(90^\circ-15^\circ)}{\sin(90^\circ-15^\circ)}$$

$$= \frac{1-\sin 15^\circ}{\cos 15^\circ} = \frac{1-\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$$

Example – 6: If sin A = K sin B, prove that $\tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\tan\frac{1}{2}(A+B)$

Solution : Given sin A = K sin B

$$\Rightarrow \frac{\sin A}{\sin B} = \frac{K}{1}$$

Using componendo & dividendo

$$\frac{\sin A+\sin B}{\sin A-\sin B} = \frac{K+1}{K-1}$$

$$\Rightarrow \frac{2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}.\sin\frac{A-B}{2}} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2}.\cot\frac{A-B}{2} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2} = \frac{K+1}{K-1}.\tan\frac{A-B}{2}$$

$$\Rightarrow \tan\frac{A-B}{2} = \frac{K-1}{K+1}\tan\frac{A+B}{2}$$

∴ L.H.S. = R.H.S. (Proved)

Example – 7: If (1 – e) $\tan^2\frac{\beta}{2}$ = (1 + e) $\tan^2\frac{\alpha}{2}$, Prove that $\cos\beta = \frac{\cos\alpha - e}{1-e\cos\alpha}$

Solution : (1 – e) $\tan^2\frac{\beta}{2}$ = (1 + e) $\tan^2\frac{\alpha}{2}$ (Given)

$$\tan^2\frac{\beta}{2} = \frac{1+e}{1-e}\tan^2\frac{\alpha}{2}$$

L.H.S = cos β

$$= \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}$$

$$= \frac{1 - e - \tan^2\frac{\alpha}{2} - e\tan^2\frac{\alpha}{2}}{1 - e + \tan^2\frac{\alpha}{2} + e\tan^2\frac{\alpha}{2}} = \frac{\left(1 - \tan^2\frac{\alpha}{2}\right) - e\left(1 + \tan^2\frac{\alpha}{2}\right)}{\left(1 + \tan^2\frac{\alpha}{2}\right) - e\left(1 - \tan^2\frac{\alpha}{2}\right)}$$

$$= \frac{\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}{\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}$$

$$= \frac{\cos\alpha - e}{1 - e\cos\alpha} = \text{R.H.S (Proved)}$$

Example – 8: If A + B + C = π, then Prove the following

(i) sin 2A + sin 2B + sin 2C = 4 sin A. sin B. sin C

Solution : L.H.S. = sin 2A + sin 2B + sin 2C

= 2 sin (A + B) . cos (A – B) + 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) + 2 sin C . cos C

= 2 sin C . cos (A – B) + 2 sin C. cos C

= 2 sin C [cos (A – B) + cos C]

= 2 sin C [cos (A – B) – cos (A + B)]

= 2 sin C . 2 sin A . sin B

= 4 sin A . sin B . sin C R.H.S. (Proved)

(ii) sin 2A + sin 2B – sin 2C = 4 cos A . cos B. sin C

Solution : L.H.S. = sin 2A + sin 2B – sin 2C

= 2 sin (A + B) . cos (A – B) – 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) – 2 sin C. cos C

= 2 sin C. cos (A – B) – 2 sin C. cos C

= 2 sin C [cos (A – B) – cos {π – (A + B)}]

= 2 sin C {cos (A – B) + cos (A + B)}

$$= 2\sin C\left\{2\cos\frac{A - B + A + B}{2}.\cos\frac{A - B - A - B}{2}\right\}$$

= 4 sin C. cos A. cos B.

(iii) $\mathbf{\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\ \sin\frac{B}{2}\cos\frac{C}{2}}$

Solution : L.H.S. = sin A + sin B – sin C

$$= 2\sin\frac{A + B}{2}.\cos\frac{A - B}{2} - 2\sin\frac{C}{2}.\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}.\cos\frac{A - B}{2} - 2\sin\frac{C}{2}.\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\frac{C}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{(-2)\sin\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right) . \sin\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right\}$$

$$= -4\cos\frac{C}{2} . \sin\frac{A}{2} . \sin\left(-\frac{B}{2}\right)$$

$$= 4\cos\frac{C}{2} . \sin\frac{A}{2} . \sin\frac{B}{2} = \text{R.H.S (Proved)}$$

ASSIGNMENT

1. If $\tan\alpha = \frac{1}{2}, \tan\beta = \frac{1}{3}$, then find the value of $(\alpha + \beta)$

2. Find the value of $\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ}$

3. Prove that $\frac{1}{\tan 3A - \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 2A$

4. If A + B = 45º, show that (1 + tanA) (1+ tanB) = 2

5. If $(1 - e)\ \tan^2\frac{\beta}{2} = (1+e)\ \tan^2\frac{\alpha}{2}$

Prove that $\cos\beta = \frac{\cos\alpha - e}{1 - e\cos\alpha}$

6. If A + B + C = π, prove that
cos2A + cos2B + cos 2C + 1 + 4 cos A.cosB.cosC = 0

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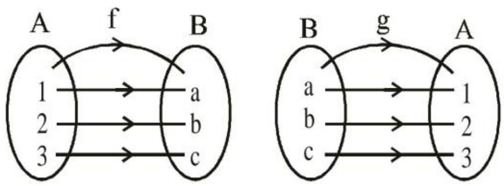
INVERSE TRIGONOMETRIC FUNCTIONS

INVERSE FUNCTION :

If $f : A \to B$ be a bijective function or one to one onto function from set A to the set B. As the function is 1 – 1, every element of A is associated with a unique element of B. As the function is onto, there is no element of B which in not associated with any element of A. Now if we consider a function g from B to A, we have for $f \in B$ there is unique $x \in A$. This g is called inverse function of f and is denoted by f^{-1}.



INVERSE TRIGONOMETRIC FUNCTION :

We know the equation x = siny means that y is the angle whose sine value is x then we have $y = \sin^{-1}x$ similarly $y = \cos^{-1}x$ if x = cosy and $y = \tan^{-1}x$ is x = tan y etc.

The function $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$, $\cot^{-1}x$ are called inverse trigonometric function.

* Properties of inverse trigonometric function.

I. Self adjusting property :

(i) $\sin^{-1}(\sin\theta) = \theta$

(ii) $\cos^{-1}(\cos\theta) = \theta$

(iii) $\tan^{-1}(\tan\theta) = \theta$

Proof:

(i) Let $\sin\theta = x$, then $\theta = \sin^{-1}x$

$\therefore \sin^{-1}(\sin\theta) = \sin^{-1}x = \theta$

proofs of (ii) * (iii) as above.

II. Reciprocal Property :

(i) $\text{cosec}^{-1}\frac{1}{x} = \sin^{-1}x$

(ii) $\sec^{-1}\frac{1}{x} = \cos^{-1}x$

(iii) $\cot^{-1}\frac{1}{x} = \tan^{-1}x$

Proof :

(i) Let $x = \sin\theta$ then $\operatorname{cosec}\theta = \dfrac{1}{x}$

so that $\theta = \sin^{-1}x$ & $\theta = \operatorname{cosec}^{-1}\dfrac{1}{x}$

$\therefore\ \sin^{-1}x = \operatorname{cosec}^{-1}\dfrac{1}{x}$

(ii) and (iii) may be proved similarly

III. Conversion property :

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

(ii) $\cos^{-1}x\ \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$

Proof:

(i) Let $\theta = \sin^{-1}x$ so that $\sin\theta = x$

Now $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

i.e., $\theta = \cos^{-1}\sqrt{1-x^2}$

Also $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{x}{\sqrt{1-x^2}}$

$\Rightarrow \theta = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

Thus we have $\theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

Theorem – 1 : Prove that

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$

(iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \dfrac{\pi}{2}$

Proof :

(i) Let $\sin^{-1}x = \theta$, then

$x = \sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$

$\Rightarrow\ \cos^{-1}x = \dfrac{\pi}{2} - \theta = \dfrac{\pi}{2} - \sin^{-1}x$

$\Rightarrow\ \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

(ii) and (iii) can be proved similarly.

Theorem – 2 : If xy < 1, then

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Proof : Let $\tan^{-1}x = \theta_1$ and $\tan^{-1} y = \theta_2$

Then

$\tan\theta_1 = x$ and $\tan\theta_2 = y$

$$\Rightarrow \quad \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{x+y}{1-xy}$$

$$\Rightarrow \quad \theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Theorem – 3 : $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Proof : Let $\tan^{-1}x = \theta$, and $\tan^{-1}y = \theta_2$

$$\Rightarrow \quad \tan\theta_1 = x \text{ and } \tan\theta_2 = y$$

$$\Rightarrow \quad \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1, \tan\theta_2} = \frac{x-y}{1+xy}$$

$$\Rightarrow \quad \theta_1 - \theta_2 = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \quad \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

Note : $\tan^{-1} + \tan^{-1}y + \tan^{-1}z$

$$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Theorem – 4 : Prove that :

(i) $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

Proof :

(i) Let $\sin^{-1}x = \theta$, Then, $x = \sin\theta$

$$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta.\ \sqrt{1-\sin^2\theta}$$

$$= 2x\sqrt{1-x^2}$$

$$\Rightarrow \quad 2\theta = \sin^{-1}\left[2x\sqrt{1-x^2}\right] \quad \Rightarrow \quad 2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$$

(ii) Let $\cos^{-1}x = \theta$ Then, $x = \cos\theta$

$$\therefore \quad \cos 2\theta = (2\cos^2\theta - 1) = 2x^2 - 1$$

$$\Rightarrow \quad 2\theta = \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow \quad 2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$$

Theorem – 5 : Prove that

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{(1-x^2)(1-y^2)}\right]$

(iii) $\sin^{-1}x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{(1-x^2)(1-y^2)}\right]$

Proof :

(i) Let $\sin^{-1}x = \theta_1$, and $\sin^{-1}y = \theta_2$, Then

$\sin \theta_1 = x$ and $\sin \theta_2 = y$

$\therefore \quad \sin (\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$

$= \sin \theta_1 \sqrt{1-\sin^2\theta_2} + \sqrt{(1-\sin^2\theta_1)} \sin\theta_2$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad \theta_1 + \theta_2 = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

The other results may be proved similarly.

Example – 1 : **If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$**

Solution : Given $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = (\pi - \cos^{-1}z)$

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos(\pi - \cos^{-1}z)$

$\Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\cos(\cos^{-1}z) = -z$

$\Rightarrow \quad xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow \quad (xy + z)^2 = (1 - x^2)(1 - y^2) = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$ (Proved)

Example – 2 : **Find the value of $\cos \tan^{-1} \cot \cos^{-1} \frac{\sqrt{3}}{2}$**

Solution : $\cos^{-1}\frac{\sqrt{3}}{2} = \theta \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6} \Rightarrow \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\therefore \quad \cos \tan^{-1} \cot \cos^{-1}\frac{\sqrt{3}}{2} = \cos \tan^{-1} \cot \frac{\pi}{6}$

$= \cos \tan^{-1}\sqrt{3}\left[\because \tan^{-1}\sqrt{3} = \frac{\pi}{3}\right] = \cos\frac{\pi}{3} = \frac{1}{2}$

Example – 3 : Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$.

Solution : L.H.S $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} \qquad \left(\because 2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2}\right)$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{1}{2}+\frac{1}{7}}{1-\frac{1}{2}\times\frac{1}{7}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{9}{14}}{\frac{13}{14}} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{9}{13}$$

$$= \tan^{-1}\frac{\frac{1}{2}+\frac{9}{13}}{1-\frac{1}{2}\times\frac{9}{13}} = \tan^{-1}\frac{\frac{31}{26}}{\frac{17}{26}} = \tan^{-1}\frac{31}{17} = \text{R.H.S. (Proved)}$$

Example – 4 : Prove that $\cot^{-1}9 + \text{cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$

Solution : L.H.S. $= \cot^{-1}9 + \text{cosec}^{-1}\frac{\sqrt{41}}{4}$

$$= \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} \qquad \left[\because \text{cosec}^{-1}\frac{\sqrt{41}}{4} = \tan^{-1}\frac{4}{5}\right]$$

$$= \tan^{-1}\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\cdot\frac{4}{5}} = \tan^{-1}\frac{\frac{5+36}{45}}{\frac{45-4}{45}} = \tan^{-1}\frac{\frac{41}{45}}{\frac{41}{45}}$$

$$= \tan^{-1}1 = \frac{\pi}{4} \text{ R.H.S. (Proved)}$$

ASSIGNMENT

1. Find the value of $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

2. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$. Show that

$$x.\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

3. If s $\sin^{-1}\frac{x}{5} + \operatorname{cos}ec^{-1}\frac{5}{4} = \frac{\pi}{2}$. Find the value of x.

❒ ☸ ❒

CHAPTER - 2

DIFFERENTIAL CALCULUS

1.LIMIT OF A FUNCTION

Lets discuss what a function is

A function is basically a rule which associates an element with another element.

There are different rules that govern different phenomena or happenings in our day to day life.

For example,

i. Water flows from a higher altitude to a lower altitude
ii. Heat flows from higher temperature to a lower temperature.
iii. External force results in change state of a body(Newton's 1st Rule of motion) etc.

All these rules associates an event or element to another event or element, say , x with y.

Mathematically we write,

$$y = f(x)$$

i.e. given the value of x we can determine the value of y by applying the rule 'f'

for example,

$$y = x + 1$$

i.e we calculate the value of y by adding 1 to value of x. This is the rule or function we are discussing.

Since we say a function associates two elements, x and y we can think of two sets A and B such that x is taken from set A and y is taken from set B. Symbolically we write

x ε A (x belongs to A)

y ε B (x belongs to B)

y = f(x) can also be written as

(x,y) ε f

Since (x,y) represents a pair of elements we can think of these in relations

f $\subseteq$ A X B or

f can thought of as a sub set of the product of sets A and B we have earlier referred to.

And, therefore, the elements of f are pair of elements like (x,y).

In the discussion of a function we must consider all the elements of set A and see that no x is associated with two different values of y in the set B

What is domain of function

Since function associates elements x of A to elements y of B and function must take care of all the elements of set A we call the set A as domain of the function. We must take note of the fact that if the function can not be defined for some elements of set A , the domain of the function will be a subset of A.

Example 1

Let A = $\{1,2,3,4,-1,0,-4\}$

B = $\{0,1,2,3,4,-1,-2,-3\}$

The function is given by

y = f(x) = x + 1

for x=1, y= 2

x=2, y=3

x=3,y=4

x=4,y=5

x=-1,y=0

x=0,y=1

x=-4,y=-3

clearly y=5 and y= -3 do not belong to set B. therefore we say the domain of this function is

the set $\{0,1,2,3,-1,\}$ which is a sub set of set A.

What is range of a function

Range of the function is the set of all y's whose values are calculated by taking all the values of x in the domain of the function. Since the domain of the function is either is equal to A or sub set of set A, range of the function is either equal to set b or sub set of set B.

In the earlier example,

Range of function is the set $\{1,2,3,4,0\}$ which is a sub set of set B

SOME FUNDAMENTAL FUNCTIONS

Constant Function

Y = f(x)=K, for all x

The rule here is: the value of y is always k, irrespective of the value of x

This is a very simple rule in the sense that evaluation of the value of y is not required as it is already given as k

Domain of 'f' is set of all real numbers

Range of 'f' is the singleton set containing 'k' alone.

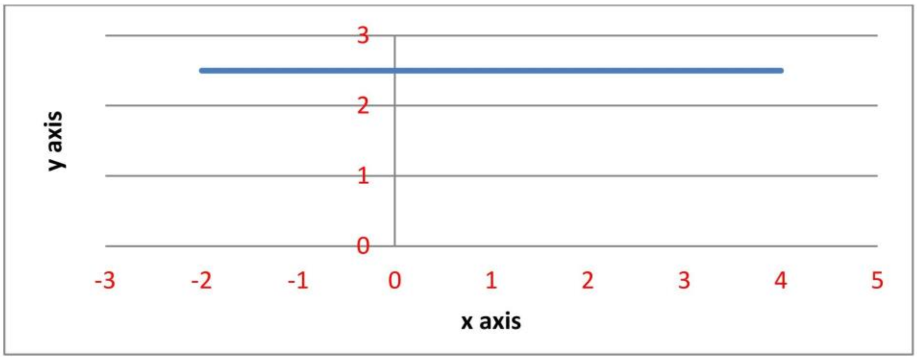
Or

Dom= R, set of all real numbers

Range= {k}

Graph of Constant Function

Let y = f(x) = k =2.5



The graph is a line parallel to axis of x

Identity Function

Y = f(x)=x, for all x

The rule here is: the value of y is always equals to x

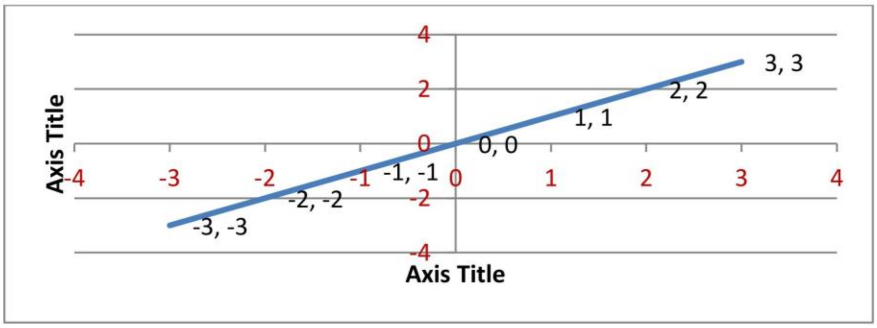
This is also a very simple rule in the sense that the value of y is identical with the value of x saving our time to calculate the value of y.

Dom = R

Range = R

i.e. Domain of the function is same as Range of the function

Graph of Identity Function



Modulus Function

$$y = f(x) = |x| = \begin{Bmatrix} x, x \geq 0 \\ -x, x < 0 \end{Bmatrix}$$

The rule here is: the value of y is always equals to the numerical value of x, not taking in to consideration the sign of x.

Example

Y =f(2)=2

Y=f(0)=0

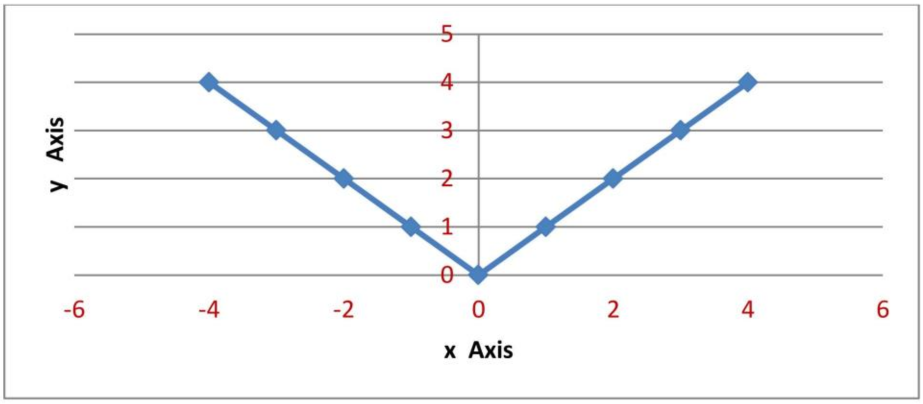
Y=f(-3)=3

This function is usually useful in dealing with values which are always positive for example, length, area etc.

Dom = R

Range = R^+ U $\{0\}$

Graph of Modulus Function



Signum Function

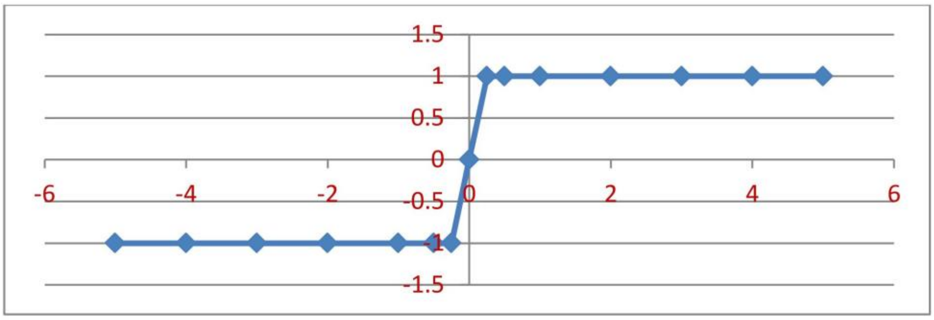
$$y = f(x) = \begin{Bmatrix} \dfrac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{Bmatrix} = \begin{Bmatrix} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{Bmatrix}$$

This is also a very simple rule in the sense that the value of y is 1 if x is positive , 0 when x=0, and -1 when x is negative.

Dom = R

Range = $\{-1,0,1\}$

Graph of Signum Function



Greatest Integer Function

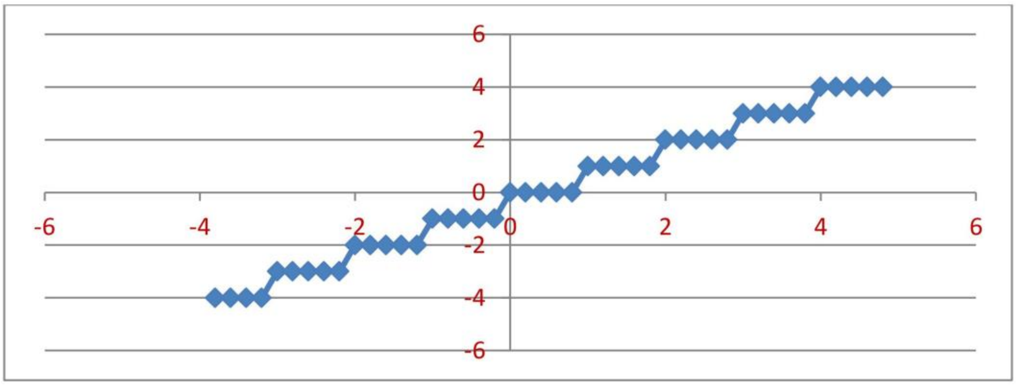
$$y = f(x) = [x] = greatest\ integer \leq x$$

For Example $[0] = 0, [0.2] = 0, [2.5] = 2, [-3.8] = -4\ , etc.$

Dom = R

Range=Z(set of all Integers)

Graph of The function



Exponential Function

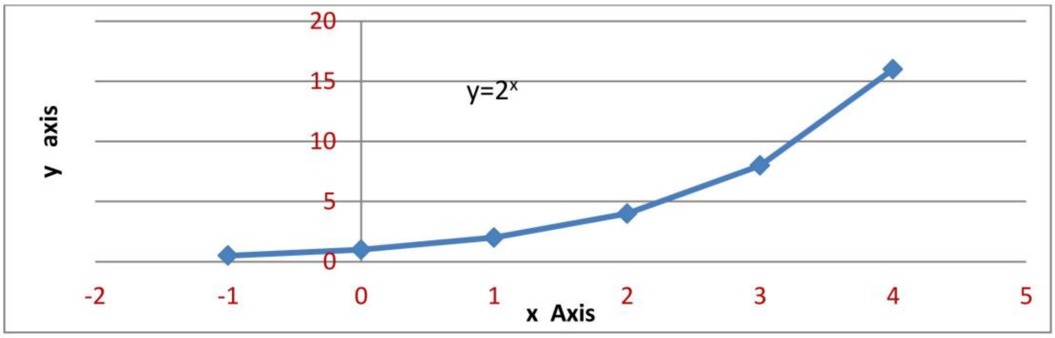
$$y = f(x) = a^x\ where\ a\ > 0$$

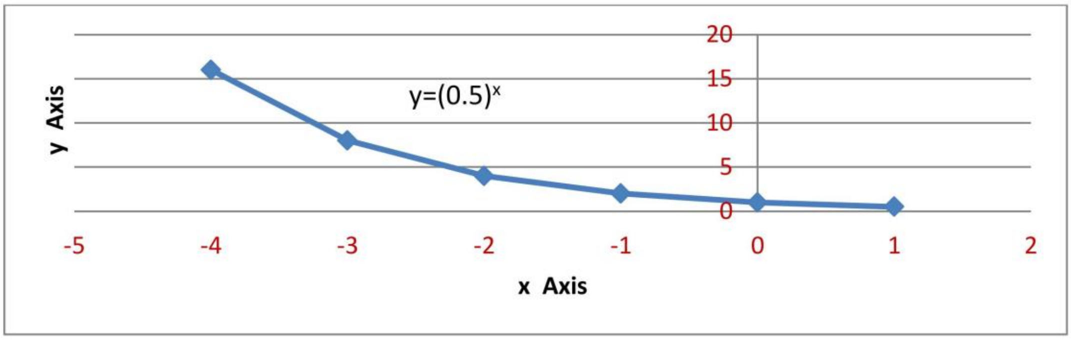
Dom = R

Range= R^+

The specialty of the function is that whatever the value of x, y can never be 0 or negative

Graph of Exponential Function





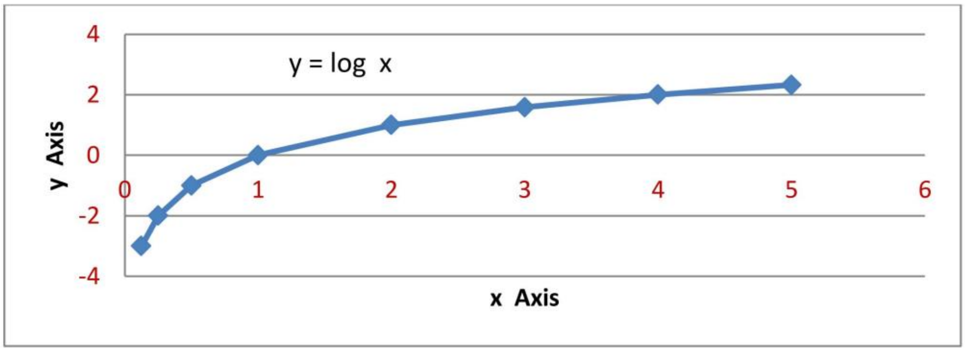
Logarithmic Function

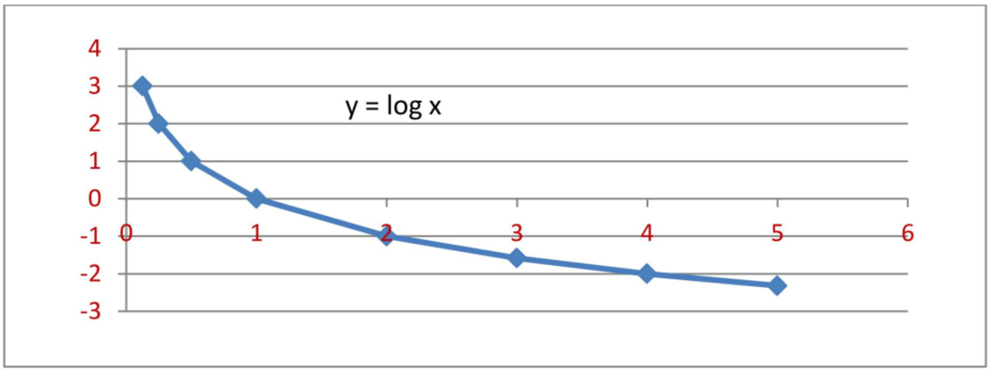
$$y = f(x) = \log_a x$$

Dom = R^+

Range =

Graph of Logarthmic Function




y = log x
4
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6

Differentiation

A function f(x) is said to be differentiable at a point x=c iff

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \quad exists$$

In general, a function is differentiable iff

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad exists$$

Once this limit exists, it is called the differential coefficient of f(x) or the derivative of the function f(x) at x=c

Or

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$$

Where $f'(c)$ and $f'(x)$ are the differential coefficient or the derivative of the function, the first being defined at x=c

Examples

Consider the function

$$y=f(x)=k \text{ or the constant function}$$

In this case the differential coefficient $f'(x)$is given by

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{k-k}{\delta x}=0$$

Therefore the constant function is differentiable everywhere and the derivative is zero

Consider the function

$$y = f(x) = x^2$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{(x+\delta x) - x}$$

$$= 2x$$

Consider the function

$$y = f(x) = \sin x$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{sin(x+\delta x) \quad - sinx}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2cos\left(\frac{x+\delta x + x}{2}\right) \quad \times sin\left(\frac{x+\delta x - x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{cos\left(\frac{x+\delta x + x}{2}\right) \quad \times sin\left(\frac{x+\delta x - x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \frac{cos\left(\frac{x+\delta x + x}{2}\right) \quad \times sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \lim_{\delta x \to 0} cos\left(x + \frac{\delta x}{2}\right) \times 1$$

$$= \cos x$$

Therefore

$$y = f(x) = sinx$$

$$\frac{dy}{dx} = cosx$$

Algebra of derivatives

Consider two differentiable functions u(x) and v(x)

Let

$$y = u + v$$

Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Let

$$y = u \times v$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let

$$y = \frac{u}{v}, \qquad v \neq 0$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example

1

$$y = sinx + x^3$$

$$\frac{dy}{dx} = cosx + 2x$$

2

$$y = x^2cosx$$

$$\frac{dy}{dx} = x^2(-sinx) + cosx(2x)$$

$$= -x^2 sinx + 2xcosx$$

3

$$y = \frac{sinx}{cosx}$$

$$\frac{dy}{dx} = \frac{cosxcosx - sinx(-sinx)}{(cosx)^2}$$

$$\frac{dy}{dx} = \frac{(cosx)^2 + (sinx)^2}{(cosx)^2}$$

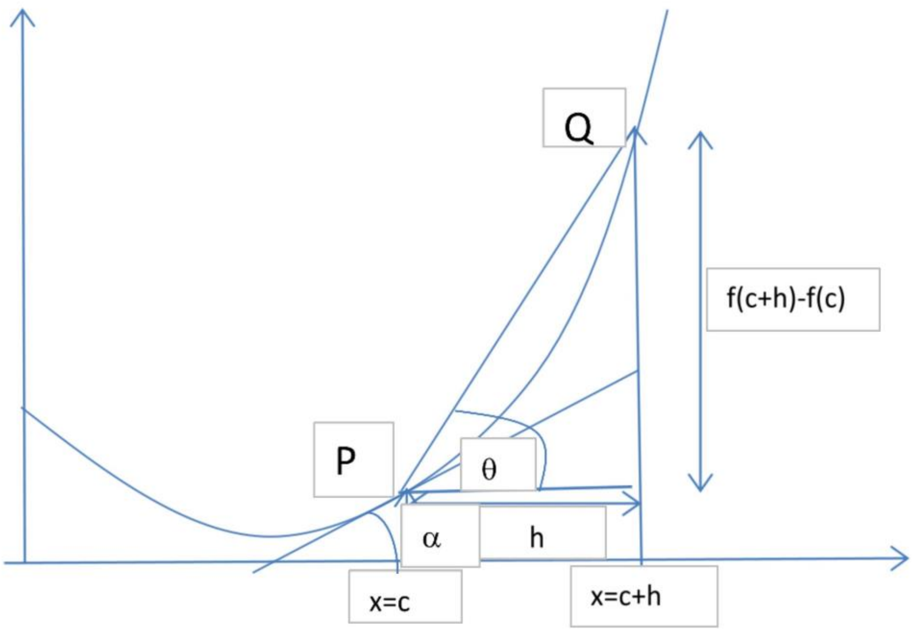
$$\frac{dy}{dx} = \frac{(cosx)^2 + (sinx)^2}{(cosx)^2}$$

$$\frac{dy}{dx} = \frac{1}{(cosx)^2} = (secx)^2$$

Geometrical meaning of $f'(c)$

Consider the graph of a function

$$y = f(x)$$



$$\frac{f(c+h)-f(c)}{h}$$

Represents the ratio of height to base of the angle the line joining the point P(c,f(c)) and Q (c+h,f(c+h))

i.e

$$\frac{f(c+h)-f(c)}{h} = tan\theta$$

Where θ is the angle the line joining the point P and Q makes with the positive direction of x axis.

In the limiting case as h→ 0 i.e as Q→ P the line PQ becomes the tangent line and the the angle $\theta\ becomes\ the\ angle\ \alpha$ which the tangent line makes with the positive direction of x axis

i.e

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = f'(c) = tan\alpha = m\ (the\ slope\ of\ the\ tangent)$$

Application to Geometry

To find the equation of the tangent line to the curve y=f(x) at x=x_0

The equation of line passing through the point $(x_0, f(x_0))$ is give by

$$y - f(x_0) = m(x - x_0)$$

Where ' m' is the slope of the tangent line.

As, we have seen

$$m = f'(x_0)$$

The equation is therefore

$$y - f(x_0) = f'(x_0)(x - x_0)$$

In the above example if we take

$f(x) = x^2\ and\ the\ point\ x_0 = 1$

The equation to the tangent at the point is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Or

$$y - 1^2 = 2 \times 1(x - 1)$$

where

$$f(x_0) = x_0^2 = 1^2 \; and \; f'(x_0) = 2 \times x_0 = 2 \times 1$$

i.e

the equation is

$$y - 1 = 2(x - 1)$$

Derivative as rate measurer

Remember the definition

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

The quantity

$$\frac{f(c+h) - f(c)}{h}$$

$$measures\ the\ rate\ of\ change\ in\ f(c) wth\ respect\ to\ change\ h\ in\quad 'c'$$

Consider the linear motion of a particle given as

$$s = f(t)$$

Where ‘s’ denotes the distance traversed and ‘t’ denotes the time taken

The ratio

$$\frac{s}{t}$$

Denotes the **average velocity** of the particle

To calculate the local velocity or instantaneous velocity at a point of time t=$t_0$we proceed in the following way

Consider an infinitesimal distance ’ $\delta s'$ traversed from time t=t_0 in time ’ $\delta t'$

The ratio

$$\frac{\delta s}{\delta t}$$

Still represents a average value of the velocity

The instantaneous velocity at t=t_0 can be calculated by considering the following limit

$$\lim_{\delta t \to 0} \frac{\delta s}{\delta t}$$

or

$$v = \frac{ds}{dt}$$

Where ‘v’ represents the instantaneous ***velocity*** which is defined as rate of change of displacement

Similarly, we can write the mathematical expression for ***acceleration***

As

$$a = \frac{dv}{dt}$$

Or the rate of change of velocity

Example

If the motion of a particle is given by

$$s = f(t) = 2t + 5$$

Which is linear in nature, we can calculate velocity at t=3

$$v(t = 3) = \frac{ds}{dt} = 2$$

It is clear that the velocity is independent of time ‘t’.

i.e

the above motion has constant or uniform velocity.

And, therefore, the acceleration

$$a = \frac{dv}{dt} = 0$$

Or the motion does not produce any acceleration.

Consider another motion of a particle given as

$$s = f(t) = 2t^2 + 3$$

Here the velocity at t=3 can be calculated as

$$v(t = 3) = \frac{ds}{dt} = 4t = 4 \times 3 = 12$$

And the acceleration

$$a = \frac{dv}{dt} = 4$$

Therefore we can say that the motion is said to have constant or uniform acceleration

Derivatives of implicit function

Consider the equation of a circle

$$x^2 + y^2 = r^2$$

This is an implicit function

Lets differentiate this equation with respect x throughout, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Derivative of parametric function

The equation of a circle can also be written as

$$x = rcost$$

$$y = rsint$$

This is called parametric function having parameter 't'

In this case

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{rcost}{-rsint} = \frac{x}{-y} = \frac{-x}{y}$$

Derivative of function with respect to another function

Consider the functions

$$y = f(x)$$

$$z = g(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{g'(x)}$$

Example

Let

$$y = sin(x)$$

$$z = x^3$$

$$\frac{dy}{dx} = \frac{f'(x)}{g'(x)} = \frac{cosx}{3x^2}$$

Derivative of composite function

Consider the function

$$y = f(u) \ where \ u = g(x)$$

Then y is called a composite function

In this case

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is called Chain Rule. This can be extended to any number of functions.

Example

1.Let

$$y = sinx^2$$

This can be written as

$$y = sinu$$

And

$$u = x^2$$

Applying chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = cosu \times 2x = 2xcosx^2$$

2.Let

$$y = tane^{x^2}$$

This can be written as

$$y = tanu$$

And

$$u = e^v$$

$$v = x^2$$

Applying chain rule, we have

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}=sec^2u\times e^v\times 2x=sec^2e^{x^2}\times e^{x^2}\times 2x$$

Derivatives of inverse function

$$since\quad \frac{\delta x}{\delta y}=\frac{1}{\frac{\delta y}{\delta x}}$$

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$$

As $\delta x\to 0, \delta y\ also\to 0$

Which follows from the fact that

$$y=f(x)\ being\ a\ differentiable\ function\ is\ a\ continuous\ function$$

And the condition of continuity guarantees the above fact.

Derivative of inverse trigonometric function

Let

$$y=sin^{-1}x$$

Where $y\epsilon\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

This can be written as

$$x=siny$$

$$\frac{dx}{dy}=cosy$$

Or

$$\frac{dy}{dx}=\frac{1}{cosy}=\frac{1}{\mp\sqrt{(1-sin^2y)}}=\frac{1}{\mp\sqrt{(1-x^2)}}=\frac{1}{\sqrt{(1-x^2)}}$$

Since cosy is positive in the domain

Let

$$y = cos^{-1}x$$

Where $y\epsilon(0,\pi)$

This can be written as

$$x = cosy$$

$$\frac{dx}{dy} = -siny$$

Or

$$\frac{dy}{dx} = \frac{-1}{siny} = \frac{-1}{\mp\sqrt{(1-cos^2y)}} = \frac{-1}{\mp\sqrt{(1-x^2)}} = \frac{-1}{\sqrt{(1-x^2)}}$$

Since sin y is positive in the domain

Let

$$y = sec^{-1}x$$

Where $y\epsilon\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$

This can be written as

$$x = secy$$

$$\frac{dx}{dy} = secy \times tany$$

Or

$$\frac{dy}{dx} = \frac{1}{secy \times tany} = \frac{1}{x\sqrt{(sec^2y-1)}} = \frac{1}{x(\mp\sqrt{(x^2-1)})} = \frac{1}{|x|\sqrt{(1-x^2)}}$$

Since $secy \times tany$ is positive in the domain

Let

$$y = cosec^{-1}x$$

Where $y\epsilon\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$

This can be written as

$$x = cosecy$$

$$\frac{dx}{dy} = -cosecy \times coty$$

Or

$$\frac{dy}{dx} = \frac{-1}{cosecy \times coty} = \frac{-1}{x\sqrt{(cosec^2y-1)}} = \frac{-1}{x(\mp\sqrt{(x^2-1)})} = \frac{-1}{|x|\sqrt{(1-x^2)}}$$

Since $cosecy \times coty$ is positive in the domain

Let

$$y = tan^{-1}x$$

This can be written as

$$x = tany$$

$$\frac{dx}{dy} = sec^2y$$

Or

$$\frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1+tan^2y} = \frac{1}{1+x^2}$$

Let

$$y = cot^{-1}x$$

This can be written as

$$x = coty$$

$$\frac{dx}{dy} = -cosec^2y$$

Or

$$\frac{dy}{dx} = \frac{-1}{cosec^2 y} = \frac{-1}{1 + cot^2 y} = \frac{-1}{1 + x^2}$$

Higher order derivatives

Let

$$y = f(x)$$

Is differentiable and also

$$\frac{dy}{dx} = f'(x)$$

Is differentiable. Then we define

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

$$=$$

$$= \lim_{\delta x \to 0} \frac{f'(x + \delta x) - f'(x)}{\delta x}$$

This is the 2^{nd}. Order derivative of the function

Similarly we can define higher order derivatives of the function

Example

Let

$$y = f(x) = x^3 + x^2 + x + 1$$

$$\frac{dy}{dx} = f'(x) = 3x^2 + 2x + 1$$

$$\frac{d^2y}{dx^2} = f''(x) = 6x + 2$$

Consider the Function

$$y = f(x) = Acosx + Bsinx$$

Here

$$\frac{dy}{dx} = f'(x) = -Asinx + Bcosx$$

$$\frac{d^2y}{dx^2} = f''(x) = -Acosx - Bsinx = -y$$

i.e in this case

$$\frac{d^2y}{dx^2} + y = 0$$

Monotonic Function

Increasing function

Consider a function

$$y = f(x)$$

If $x_2 > x_1 \ \ implies\ f(x_2) > f(x_1)$

Then the function is increasing

Example

$$y = f(x) = x + 1$$

$$f(2) = 2 + 1 = 3$$

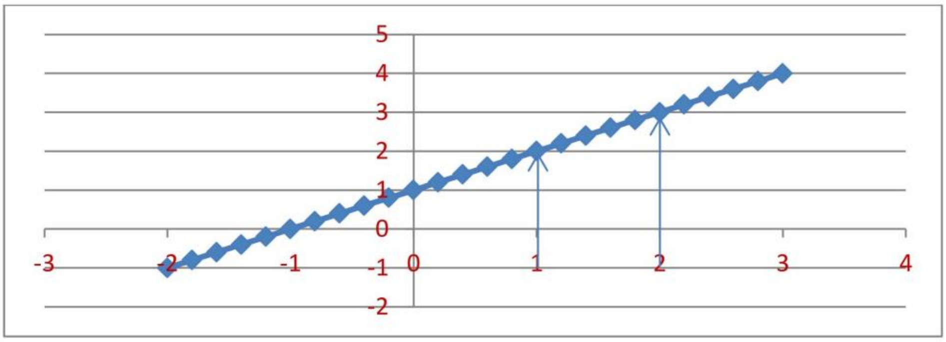
$$f(1) = 1 + 1 = 2$$

Or

$$f(2) > f(1)$$

Therefore the function is increasing

Graph of the function



Decreasing function

Consider a function

$$y = f(x)$$

If $x_2 > x_1 \ implies \ f(x_2) < f(x_1)$

Then the function is decreasing

Consider the function

$$y = f(x) = \frac{1}{x}$$

$$f(2) = \frac{1}{2}$$

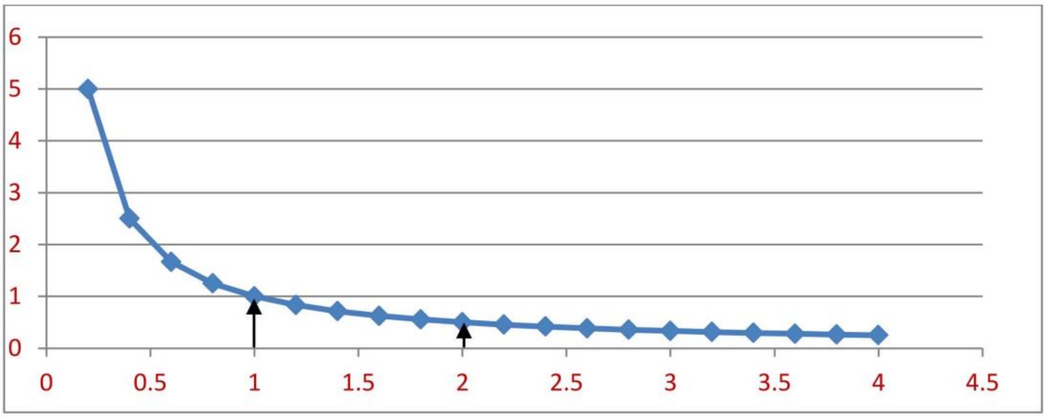
$$f(1) = \frac{1}{1} = 1$$

Or

$$f(2) < f(1)$$

Therefore the function is decreasing

Graph of the function



A function either increasing or decreasing is called monotonic.

Derivative of Increasing Function

If f(x) is increasing, then

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} > 0$$

i.e

for increasing function the derivative is always positive

Derivative of Decreasing Function

If f(x) is decreasing, then

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} < 0$$

i.e

for decreasing function the derivative is always negative

example

let

$$y = f(x) = x + 1$$

$$\frac{dy}{dx} = f'(x) = 1 > 0$$

Therefore the function is increasing

Let

$$y = f(x) = \frac{1}{x}$$

$$\frac{dy}{dx} = f'(x) = \frac{-1}{x^2} < 0$$

Therefore the function is decreasing

Let

$$y = f(x) = x^2$$

$$\frac{dy}{dx} = f'(x) = 2x$$

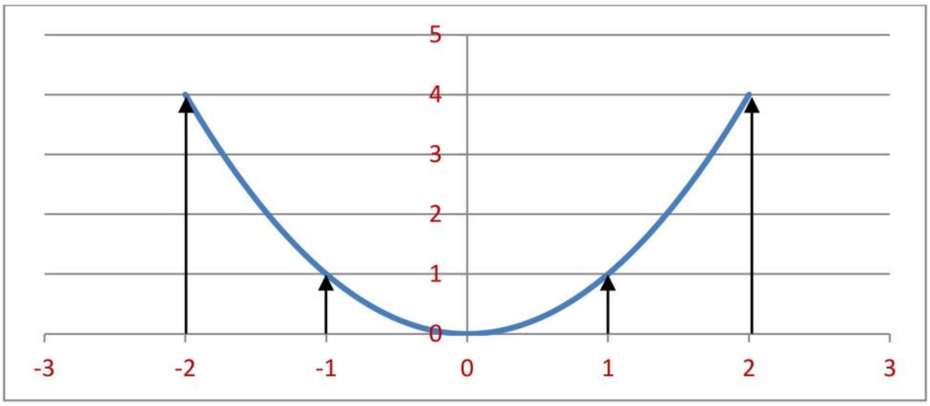
$$> 0 \; for \; x > 0$$

$$< 0 \; for \; x < 0$$

Therefore the function is increasing for $x > 0$ and decreasing for $x < 0$

Graph of the function

$$y = f(x) = x^2$$



CHAPTER - 3

ALGEBRA

COMPLEX NUMBERS

INTRODUCTION

We have the knowledge of integers, fractions and irrational number (all these constitute real numbers). But if we try to solve the equation $x^2 + 1 = 0$, we observe that these numbers are not adequate. Trying to solve this equation, we arrive at $x^2 = -1$ i.e. $x = \sqrt{-1}$.

Square of a positive real number is positive and that of a negative real is also positive. So there is no real number whose square is negative. So we are to create a new kind of number. We define the square root of a negative number as imaginary number' particularly $\sqrt{-1} = i$, the basic imaginary number.

Then $\sqrt{-4} = 2i$, $\sqrt{-2} = \sqrt{2}$ i and so on .

Imaginary numbers :

Taking $i = \sqrt{-1}$, we observe that

$i^2 = -1$

$i^3 = -1.i = -i$

$i^4 = 1$

Since $i^4 = 1$, $i = i^5 = i^9 = i^{13} = ... = i^{4n+1}$, where n is an integer.

$i^2 = i^6 = i^{10} = i^{14} = ... = i^{4n+2}$

$i^3 = i^7 = i^{11} = i^{15} = ... = i^{4n+3}$

$i^4 = i^8 = i^{12} = i^{16} = ... = i^{4n}$.

COMPLEX NUMBERS

The numbers of the form a + ib where a and b are real numbers and $i = \sqrt{-1}$, are known as complex numbers.

In complex number z = a + ib, the real numbers a and b are respectively know as real and imaginary parts of z and we write :

Re(z) = a and Im (z) = b

Thus the set C of all complex numbers is given by $C = \{z : z = a + ib, \text{ where } a, b \in R\}$

Purely real and purely imaginary numbers :

A complex number z is said to be

(i) Purely real, if Im (z) = 0

(ii) Purely imaginary, if Re (z) = 0

Thus, 2, –7, $\sqrt{3}$ etc are all purely real numbers.

While2i, $i\sqrt{3}$, $\frac{-1}{2}$ i etc are purely imaginary.

Conjugate of a complex number :

The conjugate of a complex number 'z', denoted by $\bar{z}$ is the complex number obtained by changing the sign of imaginary part of z.

e.g. $\left(\overline{2+3i}\right) = (2 - 3i)$; $\left(\overline{3+5i}\right) = (3 - 5i)$,

$\overline{6i} = -6i$; $-\overline{2i} = 2i$

Modulus of a complex number : If z = x + iy be a complex number, the modulus of z, written as | z | is a real number $\sqrt{x^2+y^2}$.

For $z = 3 + 4i$, $|z| = \sqrt{3^2+4^2} = 5$.

Also $|\bar{z}| = |z|$.

If $z = x + iy$, $\bar{z} = x - iy$.

$|z| = \sqrt{x^2+y^2}$, $|\bar{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2}$

SUM DIFFERENCE AND PRODUCT OF COMPLEX NUMBERS

For any complex number

$z_1 = (a + ib)$ and $z_2 = (c + id)$

we define

(i) $z_1 + z_2 = (a + ib) + (c + id) = [(a + c) + i\,(b + d)]$

(ii) $z_1 - z_2 = (a + ib) - (c + id) = [(a - c) + i\,(b - d)]$

(iii) $z_1 z_2 = (a + ib)(c + id) = [(ac - bd) + i\,(ad + bc)]$

CUBE ROOTS OF UNITY

Let $\sqrt[3]{1} = x$, then

$x^3 = 1$ [on cubing both sides]

$\Rightarrow x^3 - 1 = 0 \quad \Rightarrow \quad (x - 1)(x^2 + x + 1) = 0$

$\Rightarrow x - 1 = 0 \quad$ or $\quad x^2 + x + 1 = 0$

$\Rightarrow x = 1 \quad$ or $\quad x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow x = 1 \quad$ or $\quad x = \dfrac{-1 \pm i\sqrt{3}}{2}$

$\therefore$ The cube roots of unity are 1, $\dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$

Clearly one of the cube roots of unity is real and the other two are complex.

***Example – 1 :* Express in the form a + ib**

(i) $\mathbf{\dfrac{3+5i}{2-3i}}$ **(ii)** $\mathbf{\dfrac{(1+i)^2}{3-i}}$

Sol

$$\textbf{Sol}^n:\ \text{(i)}\ \frac{3+5i}{2-3i} = \frac{(3+5i)(2+3i)}{(2-3i)(2+3i)} = \frac{6+10i+9i+15i^2}{4-9i^2} = \frac{-9+19i}{13} = \frac{-9}{13} + \frac{19}{13}i$$

$$\text{(ii)}\ \frac{(1+i)^2}{3-i} = \frac{(1+i^2+2i)(3+i)}{(3-i)(3+i)} = \frac{6i-2i^2}{9-i^2} = \frac{6i+2}{10} = \frac{1}{5} + \frac{3}{5}i$$

***Example – 2 :* Find the value of $\mathbf{i^{17} + i^{20} - i^{13}}$**

***Sol^n* :** $i^{17} + i^{20} - i^{13} = i^{16}.\,i + i^{20} - i^{12}.\,i = (i^2)^8.\,i + (i^2)^{10} - (i^2)^6.\,i$

$= (-1)^8 i + (-1)^{10} - (-1)^6 i = i + 1 - i = 1$

***Example – 3 :* If 1, ω, ω² are the cube roots of unity prove that**

(a) $\mathbf{(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9}$

***Sol^n*:** L.H.S. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega^3.\,\omega)(1 - \omega^3\omega^2)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= (1 - \omega)^2 (1 - \omega^2)^2 = [(1 - \omega)(1 - \omega^2)]^2$

$= [(1 - \omega - \omega^2 + \omega^3]^2 = (2 - \omega - \omega^2)^2$

$= (2 + 1)^2 = 3^2 = 9$

Example – 4 : **Find square roots of**

(a) 3 + 4i

Sol^n : **(a)** Let $x, y \in R$, $x + iy = \sqrt{3+4i}$

$x^2 - y^2 + i\,2xy = 3 + 4i$

Equating real and imaginary parts

$x^2 - y^2 = 3$ and $2xy = 4$

$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 25$

Hence $x^2 + y^2 = \pm 5$, But since $x^2 + y^2$ is non-negative, we have

$x^2 + y^2 = 5$

$x^2 - y^2 = 3$

$2x^2 = 8$

i.e, $x^2 = 4$, i.e, $x = \pm 2$, $y^2 = 1$ i.e., $y = \pm 1$

Hence square roots of $3 + 4i = \pm(2 + i)$

Assignment

1. If w be the cube roots of unity, then prove that
   $(1 - w + w^2)^7 + (1 + w + w^2)^7 = 128$
2. Find square roots of $-5 + 12\sqrt{-1}$

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PARTIAL FRACTIONS

ALGEBRAIC FRACTIONS, PARTIAL FRACTIONS FROM A PROPER FRACTION

Polynomial :

An expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$, where n is a positive integer and $a_0, a_1, a_2 ....$ a_n are real numbers and $a_0 \neq 0$ is called a polynomial of n^{th} degree.

Rational Fraction :

The quotient of two polynomials f(x) and g(x) where $g(x) \neq 0$ is called a rational fraction.

In this section we shall be taking functions which are quotients of two polynomial functions. Such functions are called rational functions.

The functions given by algebric expression such as

$\frac{4x^3-5}{(x-2)^3.(x+2)}$, $\frac{x^2}{(x+1)(x+2)}$, $\frac{2x-5}{x^3-x^2+x-1}$ and $\frac{x^3+x+1}{x^2-1}$ etc are called rational functions. Here both the numerator and denominator are polynomial functions. There are three types of partial fractions.

1. Proper fraction.
2. Improper fraction
3. Mixed fraction.

1. ***Proper Fraction :*** If the degree of the numerator is less than the degree of the denominator, the fraction is called proper fraction for e.g. $\frac{1}{(x+1)(x+2)}$, $\frac{2x}{x^2+3x+2}$ and $\frac{x^2}{(x-1)(x-2)(x-3)}$ etc.

$N^0 < D^0$

RESOLVING A RATIONAL FUNCTION INTO PARTIAL FRACTIONS

Case – 1 : When the denominator contains non-repeated linear factors,

for each linear non-repeated factor px + q.

there is partial fraction of the form $\frac{A}{px+q}$.

If $\frac{P(x)}{Q(x)} = \frac{P(x)}{(p_1x+q_1)(p_2x+q_2)(p_3x+q_3).......(p_nx+q_n)}$ is a proper fraction,

then $\frac{P(x)}{Q(x)} = \frac{A_1}{p_1x+q_1} + \frac{A_2}{p_2x+q_2} + \frac{A_3}{p_3x+q_3}$ ..... + $\frac{A_n}{p_nx+q_n}$, where $A_1, A_2, A_3 ...... A_n$ are constants.

Example – 1: **Split $\frac{x}{(x+1)(x+2)}$ into partial fractions**

***Sol^n* :** Let $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$$= \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$\Rightarrow$ x = A(x + 2) + B(x + 1) .... (i)

Putting x + 2 = 0 i.e. x = –2

–2 = A . 0 + B(–2 + 1)

$\Rightarrow$ –2 = –B

$\Rightarrow$ B = 2

Again put x + 1 = 0

$\Rightarrow$ x = –1

$\Rightarrow$ –1 = A(–1 + 2) + B . 0

$\Rightarrow$ –1 = A $\Rightarrow$ A = –1

Putting the values of A & B we get required partial fraction

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

Case – 2: When the denominator contains repeated linear fractors, for a repeated factor like $(px + q)^r$ of the denominator there exists the sum of r partial fractions of the form.

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \frac{A_3}{(px+q)^3} + \ldots\ldots + \frac{A_r}{(px+q)^r}$$

***Example – 2 :* Resolve into partial fractions, the function $\dfrac{1}{(x-1)(x+1)^2}$**

***Sol^n* :** Let $\dfrac{1}{(x-1)(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$$= \frac{A(x+1)^2 + B(x+1)(x-1) + C(x-1)}{(x+1)^2(x-1)}$$

$\Rightarrow$ $1 = A(x + 1)^2 + B(x + 1)(x – 1) + C(x – 1)$

Putting x – 1 = 0 i.e. x = 1,

$1 = A(1 + 1)^2$ $\Rightarrow$ 4A = 1

$\Rightarrow$ $A = \dfrac{1}{4}$

Putting x + 1 = 0 $\Rightarrow$ x = –1

1 = C(–1 – 1) $\Rightarrow$ –2C = 1 $\Rightarrow$ $C = -\dfrac{1}{2}$

Equating co-efficients of highest powers of x (i.e. x^2) on both sides in equation

$1 = A(x^2 + 2x + 1) + B(x^2 – 1) + C(x – 1)$

we get 0 = A + B

i.e A = –B i.e. $B = -\dfrac{1}{4}$

Hence required partial fraction is given by

$\therefore \dfrac{1}{(x-1)(x+1)^2}$

$= \dfrac{1}{4(x-1)} - \dfrac{1}{4(x+1)} - \dfrac{1}{2}\dfrac{1}{(x+1)^2}$

Case – 3 : When the denominator contains non-repeated quadratic factors which cannot be factorised, For each quadratic non-repeated factor $ax^2 + bx + c$ of the denominator, there exists a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$

For example, $\dfrac{1}{(x^2+\alpha)(x^2+\beta)} = \dfrac{Ax+B}{x^2+\alpha} + \dfrac{Cx+D}{x^2+\beta}$

and, $\dfrac{1}{(x^2+a_1)(x^2+a_2).....(x^2+a_n)}$

$= \dfrac{A_1x+B_1}{x^2+a_1} + \dfrac{A_2x+B_2}{x^2+a_2} + ... + \dfrac{A_nx+B_n}{x^2+a_n}$

Example – 3 : **Resolve into partial fractions** $\dfrac{\mathbf{x}}{\mathbf{(x-1)(x^2+1)}}$

***Sol^n* :** Let $\dfrac{x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$= \dfrac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$

$\Rightarrow \; x = A(x^2 + 1) + (Bx + C)(x - 1)$ .... (a)

Putting $x - 1 = 0$ i.e. $x = 1$ in (a)

$1 = A(1^2 + 1) + (Bx + C) . 0$

$\Rightarrow \; 2A = 1 \; \Rightarrow \quad A = \dfrac{1}{2}$

Equating coefficients of highest powers of x on both side in (a)

$x = Ax^2 + A + Bx^2 + Cx - Bx - C$

$0 = A + B$; Equating the coefficients of x^2,

$1 = C - B$; Equating the coefficients of x.

i.e, $A = -B$ i.e. $B = -\dfrac{1}{2}$

$C - B = 1$

$C + \dfrac{1}{2} = 1 \quad \Rightarrow \quad C = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

So its required partial fraction is given by

$\dfrac{x}{(x-1)(x^2+1)} = \dfrac{1}{2(x-1)} - \dfrac{(x-1)}{2(x^2+1)}$

Case – 4 : When the denominator contains repeated quadratic factors,

For each quadratic repeated factor $(ax^2 + bx + c)^r$ of the denominator, there corresponds the sum of r partial fractions of the form.

$$= \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

For example, $\dfrac{1}{(x^2 + \alpha)(x^2 + \beta)^2} = \dfrac{Ax + B}{(x^2 + \alpha)} + \dfrac{Cx + D}{(x^2 + \beta)} + \dfrac{Ex + F}{(x^2 + \beta)^2}$

$$\frac{1}{(x^2 + \alpha)^2 (x - \beta)} = \frac{Ax + B}{x^2 + \alpha} + \frac{Cx + D}{(x^2 + \alpha)^2} + \frac{E}{x - \beta}$$

Assignment

Resolving into partial fractions

1. $\dfrac{84 + 61x - x^2}{(3x + 1)(16 - x^2)}$

2. $\dfrac{x}{(1 + x)(1 + x^2)}$

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BINOMIAL THEOREM

FACTORIAL NOTATION

Let n be a positive integer. Then the product of the numbers $1\cdot 2\cdot 3$ .... $(n-1)n$ is called factorial n, and is denoted by n ! or n !.

Thus $n! = 1\cdot 2\cdot 3 \ldots\ldots\ldots (n-1)n$

Ex : $1! = 1$

$2! = 1.2 = 2$

$3! = 1.\ 2.\ 3 = 6$

$4! = 1.\ 2.\ 3.\ 4 = 24$

$5! = 1.\ 2.\ 3.\ 4.\ 5 = 120$

Deduction : $n! = n\,(n-1)\,(n-2)\,(n-3) \,..\, 3.\ 2.\ 1.$

$= n[(n-1)\,(n-2)\,(n-3) \ldots\ldots\, 3.\ 2.\ 1.]$

$= n[(n-1)!]$

Thus $5! = 5 \times (4!)$, $3! = 3 \times (2!)$ & $2! = 2 \times (1!)$

Factorial 'n' is the product of first 'n' natural numbers.

Example – 1 : Prove that :

(i) $n\,(n-1)\,)(n-2) \ldots\ldots\ldots (n-r+1) = \dfrac{n!}{(n-r)!}$

Sol^{n} ***:*** (i) $n\,(n-1)\,(n-2) \ldots\ldots\, (n-r+1)$

$$= \frac{n(n-1)(n-2)\ldots\ldots.(n-r+1).(n-r)!}{(n-r)!}$$

[Multiplying N^r and D^r by $(n-r)!$] $= \dfrac{n!}{(n-r)!}$

PERMUTATIONS

The different arrangements which can be made out of a given number of things by taking some or all at a time, are called permutations.

Example – 1: All permutations, on arrangements made with the letters a, b, c by taking two at a time are : ab, ba, ac, ca, bc, cb.

Example – 2: All permutations made with the letters a, b, c taking all of at a time are : abc, acb, bac, bca, cab, cba.

Notations : Let r and n be positive integers. Such that $1 \leq r \leq n$

Then the number of different permutations of n dissimilar things, taken r at a time is denoted by P(n, r) or ${}^{n}P_{r}$.

$P(n, r) = n\,(n-1)\,(n-2) \ldots\ldots (n-r+1) = \dfrac{n!}{(n-r)!}$

Note 2 : The number of all permutations of n different things taken all at a time is given by $p(n, n) = n!$

We have $P(n, r) = \frac{n!}{(n-r)!} \Rightarrow \quad P(n, n) = \frac{n!}{0!}$ [Putting r = n]

$\Rightarrow \quad n! = \frac{n!}{0!} \qquad [\because \quad P(n, n) = n!]$

$\Rightarrow \quad 0! = \frac{n!}{n!} = 1$, We are now bound to define $0! = 1$,

Each of the different groups of selections which can be formed by taking some or all of numbers of objects, irrespective of their arrangements is called a combination.

Suppose we want to select two out of three persons A, B and C. We may choose AB or BC or AC.

Clearly, AB and BA represent the same selection or group but they give rise to different arrangements. Clearly in a group or selection, the orders in which the objects are arranged is immaterial.

Example – 1 : The different combinations formed of three letters a, b, c taken two at a time are ab, bc, ac.

Example – 2 : The only combination that can be formed of three letters a, b, c taken all at a time is abc.

Example – 3 : Various groups of two out of four persons A, B, C Dare : AB, AC, AD, BC, BD, CD.

BINOMIAL THEOREM

The sum of two quantities a and b (i.e a + b) is called a binomial. Raising it to different powers, we get

$(a + b)^0 = 1, (a + b)^1 = a + b,$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Observe the presence of the co-effieicnts of these expansions in the successive rows of the following triangular arrangement.

Binomial Theorem for positive integral index :

Theorem : If x and y are real numbers, then for all $n \in N$,

$(x + y)^n = C(n, 0)\, x^n + C(n, 1)\, x^{n-1}y + C(n, 2)\, x^{n-2} y^2 + \ldots\ldots + C(n, r)\, x^{n-r} y^r + \ldots\ldots + C(n, n)\, y^n$

i.e, $(x + y)^n = \sum_{r=0}^{n} C(n,r)\, x^{n-r} y^r$

Deduction from Binomial Theorem :

(i) Replacing y by –y, we get :

$(x - y)^n = C(n, 0)\, x^n - C(n, 1)\, x^{n-1} y + C(n, 2)\, x^{n-2} y^2 + \ldots\ldots + (-1)^n C(n, n)\, y^n$

i.e, $(x - y)^n = \sum_{r=0}^{n} (-1)^r . C(n,r)\, x^{n-r} y^r$

SOME OBSERVATIONS IN A BINOMIAL EXPANSION

(i) The expansion of $(x + a)^n$ contains (n + 1) terms

(ii) Since C(n, r) = C(n, n – r), It follows that C(n, 0) = C(n, n), C(n, 1) = C(n, n – 1) and so on.

So the coefficient of the terms equidistant form the beginning and the end in a binomial expansion, are equal.

(iii) Middle Terms in a Binomial Expansion :

Since the expansion of $(x + a)^n$ contains (n + 1) terms, so

(a) $\left(\frac{1}{2}n+1\right)^{th}$ terms is the middle term, when n is even.

(b) $\frac{1}{2}$ $(n + 1)^{th}$ term and $\left[\frac{1}{2}(n+1)+1\right]^{th}$ terms are the two middle terms when n is odd.

General term in a binomial expansion :

In a binomial expansion, the (r + 1)th term, i.e., t_{r+1} is taken as the general term.

(i) In the expansion of $(x + y)^n$, we have $t_{r+1} = C(n, r)x^{n-r} y^r$;

(ii) In the expansion of $(x - y)^n$, we have $t_{r+1} = (-1)^r C(n, r) x^{n-r} y^r$;

(iii) In the expansion of $(1 + x)^n$, we have $t_{r+1} = C(n, r) x^r$,

(iv) In the expansion of $(1 - x)^n$, we have $t_{r+1} = (-1)^r C(n, r) x^r$.

Example –1 : **Find the middle terms in the following :**

$$\left(2x^2 - \frac{1}{x}\right)^7$$

Sol**ⁿ **: The number of terms in the expansion is 8. Hence there are two middle terms i.e. 4th and 5th terms.

4th term $= t_4 = t_{3+1} = (-1)^3 C(7, 3)\ (2x^2)^4.\left(\frac{1}{x}\right)^3$

$= -35 \times 16 \times x^8 \times x^{-3} = -560\ x^5$

5th term $= t_5 = t_{4+1} = (-1)^4 C(7, 4)\ (2x^2)^3.\left(\frac{1}{x}\right)^4$

$= 35 \times 8 \times x^6 \times x^{-4} = 280\ x^2$

Assignment

1.Find the coefficients of x^5 in the expansion of $\left(x - \frac{1}{x}\right)^{11}$

2. Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{x^2}\right)^{12}$

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