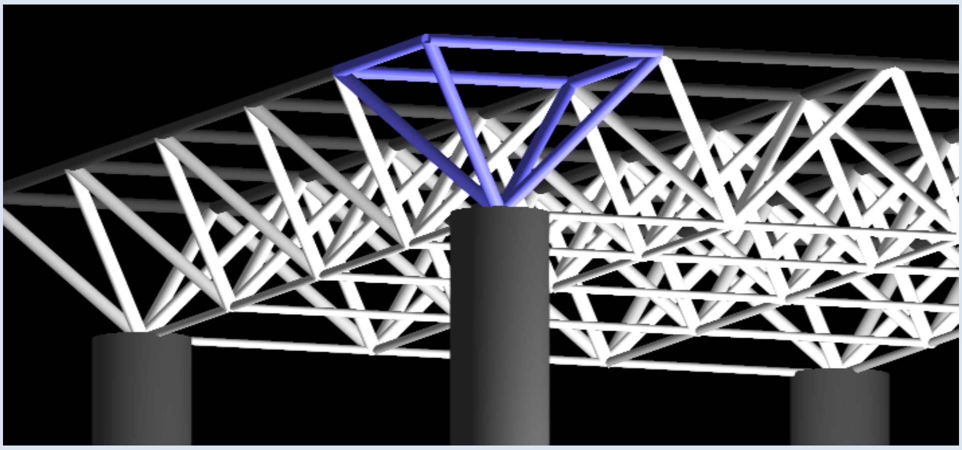
LECTURES NOTE ON

STRUCTURAL ANALYSIS



PREPARED BY

BISWAJIT BEHERA
(Lecture In Civil Engg.)

NILASAILA INSTITUTE OF SCIENCE & TECHNOLOGY

Sergarh, Balasore, Odisha

CONTENT

CHAPTER-1 :- REVIEW OF BASIC CONCEPTS

FORCE

FORCE SYSTEM

Force is that which changes or tends to change the state of rest of uniform motion of a body along a straight line. It may also deform a body changing its dimensions.The force may be broadly defined as an agent which produces or tends to produce, destroys or tends to destroy motion.Ithas a magnitude and direction.

Mathematically: **Force=Mass× Acceleration.**

Where F=force, M=mass and A=acceleration.

UNITS OF FORCE In C.GS. System: In this system, there are two units of force:

(1) Dyne and

(2) (ii) Gram force (gmf).

Dyne is the absolute unit of force in the C.G.S. system. One dyne is that force which acting on a mass of one gram produces in it anacceleration ofone centimeter per second2. In M.K.S. System: In this system, unit of force is kilogram force (kgf). One kilogram force is that force which acting on a mass of one kilogram produces in it an acceleration of 9.81 m/ sec2. In S.I. Unit: In this system, unit of force is Newton (N). One Newton is that force which acting on a mass of one kilogram produces in it an acceleration of one m /sec2. 1 Newton = 105 Dyne.

EFFECT OF FORCE

A force may produce the following effects in a body, on which it acts:

1. It may change the motion of a body. i.e. if a body is at rest, the force may set it in motion. And if the body is already in motion, the force may accelerate or decelerate it.

2. It may retard the forces, already acting on a body, thus bringing it to rest or in equilibrium.

3. It may give rise to the internal stresses in the body, on which it acts.

4. A force can change the direction of a moving object.

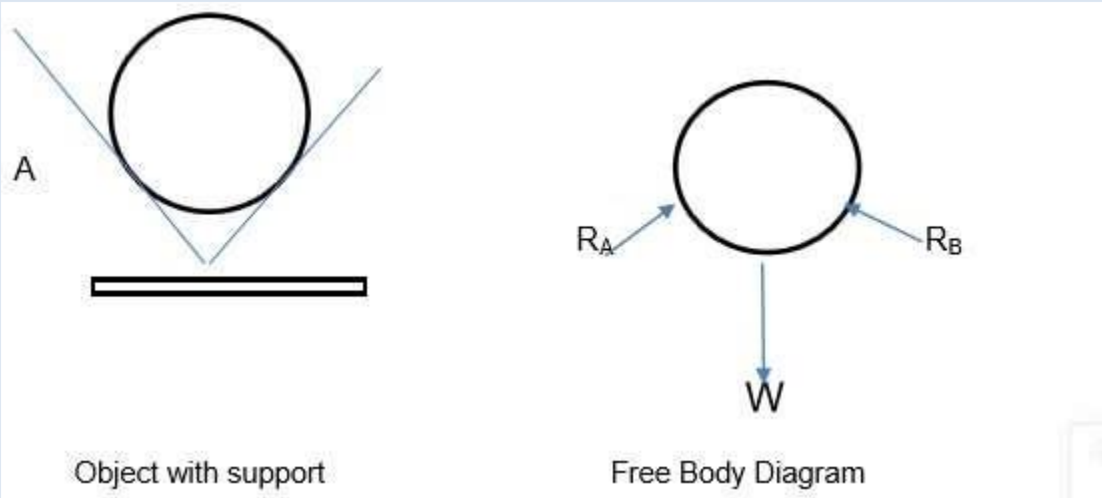
5. A force can change the shape and size of an object

SYSTEM OF FORCES

When two or more forces act on a body, they are called to form a system of forces.Force system is basically classified into following types.

- Coplanar forces
- Collinear forces
- Concurrent forces
- Coplanar concurrent forces
- Coplanar non- concurrent forces
- Non-coplanar concurrent forces
- Non- coplanar non- concurrent force

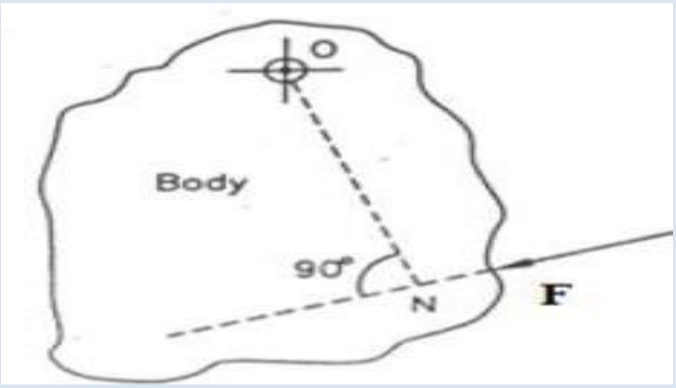
FREE BODY DIAGRAM: The representation of reaction force on the body by removing all the support or forces act from the body is called free body diagram.



MOMENT OF A FORCE

It is the turning effect produced by a force, on the body, on which it acts. The moment of a force is equal to the product of the force and the perpendicular distance of the point, about which the moment is required and the line of action of the force. Mathematically, moment, $M = P \times l$ where P = Force acting on the body, andl = Perpendicular distance between the point, about which the moment is required and the line of action of the force.

Moment of a force about a point is the product of the force and the perpendicular distance of the point from the line of action of the force.



Let a force P act on a body which is hinged at O.

Then, moment of P about the point O in the body is = F x ON,

where :ON = perpendicular distance of O from the line of action of the force F.

MOMENT OF A FORCE ABOUT AN AXIS

Let us consider a door leaf hinged to a vertical wall by several hinges. Let us consider a vertical axis XY passing through hinges as shown in Fig 1.40.

Let a force F be applied to the door leaf at right angles to its plane and at a perpendicular distance of l from the XY-axis. Then, moment of the force F about XY-axis = F x l.

UNIT OF MOMENT

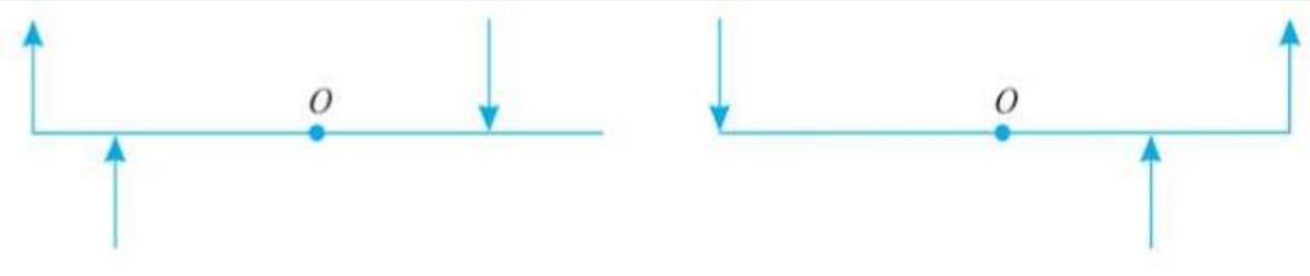
Unit of moment depends upon unit of force and unit of length.

If, however, force is measured in Newton and distance is measured in meter, the unit of moment will be Newton meter (Nm). If force is measured in kilo Newton and distance is measured in meter, unit of moment will be kilo Newton meter (kNm) and so on. Unit of moment is the same as that of work. But work is completely different from moment.

TYPES OF MOMENTS

Broadly speaking, the moments are of the following two types:

1. Clockwise moments. 2. Anticlockwise moments.



(a) Clockwise moments (b) Anticlockwise moments

PRINCIPLE OF MOMENTS

1. If a system of co-planar forces (concurrent or non-concurrent) is in equilibrium, the algebraic sum of the moments of those forces about any point in their plane is zero, i.e., the sum of the clockwise moments about any point in their plane is equal to the sum of the anticlockwise moments about the same point.

2. The algebraic sum of the moments of any number of co-planar forces (concurrent or non concurrent) about a point lying on the line of action of their resultant is zero.

3. From 1 and 2 above, it can be concluded that if the algebraic sum of the moments of any number of co-planar forces about any point in their plane is zero, either the forces are in equilibrium or their resultant passes through that point.

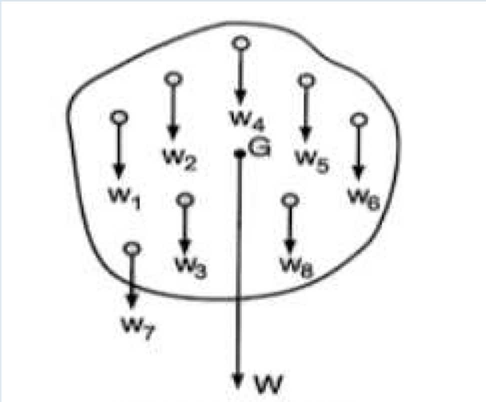
COUPLE

A pair of two equal and unlike parallel forces (i.e. forces equal in magnitude, with lines of action parallel to each other and acting in opposite directions) is known as a couple. As a matter of fact, a couple is unable to produce any translator motion (i.e., motion in a straight line).But it produces a motion of rotation in the body, on which it acts. The simplest example of a couple is the forces applied to the key of a lock, while locking or unlocking it.

CENTROID

INTRODUCTION:

A body may be considered to be made up of a number of minute particles having weights having weights w1, w2, w3,…,wn which are attracted towards the centre of body. As the particles are considered negligible in comparison to body, all the forces are considered to be parallel to each other. The resultant of all these forces acting at a point known as Centre of Gravity (C.G).



Centre of gravity

CENTRE OF GRAVITY (C.G):

Centre of Gravity of a body is a fixed point with respect to the body, through which resultant of weights of all particles of the body passes, at any plane .

CENTROID DEFINITION:

Centroid is the centre point or geometric centre of a plane figure like triangle, circle, quadrilateral, etc. The method of finding centroid is same as finding C.G of a body.

METHODS FOR CENTRE OF GRAVITY

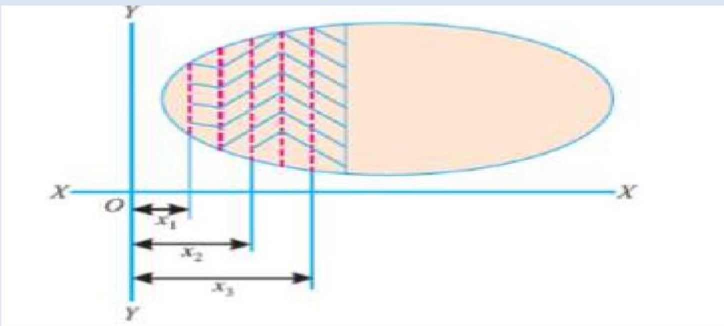
The centre of gravity (or centroid) may be found out by any one of the following two methods: 1. By geometrical considerations

2. By moments

3. By graphical method

CENTRE OF GRAVITY BY MOMENTS

Consider a body of mass M whose centre of gravity is required to be found out. Divide the body into small masses, whose centers of gravity are known as shown in Fig. 6.9. Let m1, m2, m3....; etc. be the masses of the particles and (x1, y1), (x2, y2), (x3, y3), ... be the co-ordinates of the centers of gravity from a fixed point O



Let and ̄be the co-ordinates of the centre of gravity of the body. From the principle of moments, we know that

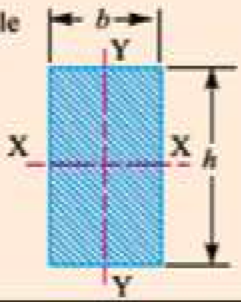
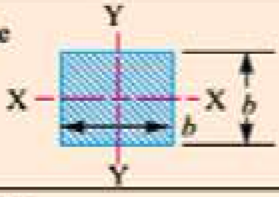
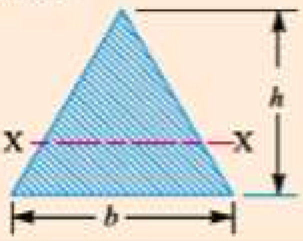
$$M\,\overline{x} = m_1 x_1 + m_2 x_2 + m_3 x_3 \,.....$$

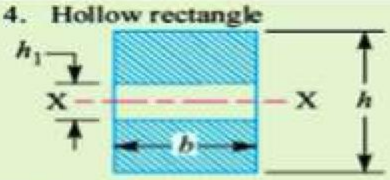
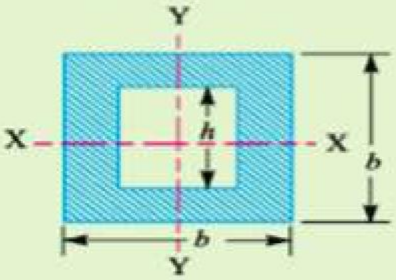
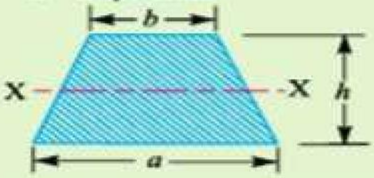
$$\overline{x} = \frac{\Sigma m x}{M}$$

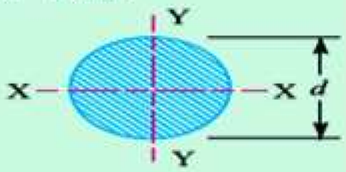
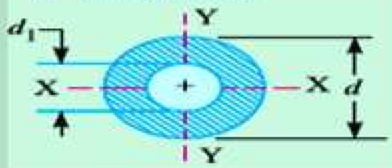
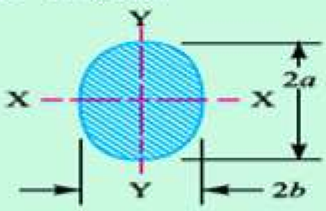
$$\overline{y} = \frac{\Sigma m y}{M},$$

$$M = m_1 + m_2 + m_3 + \,.....$$

CENTROID OF VARIOUS CROSSECTIONS

Section	*Area* (*A*)	*Moment of inertia* (*I*)	**Distance from the neutral axis to the extreme fibre (y)*
1. Rectangle 	bh	$I_{xx} = \frac{bh^3}{12}$ $I_{yy} = \frac{hb^3}{12}$	$\frac{h}{2}$ $\frac{b}{2}$
2. Square 	b^2	$I_{xx} = I_{yy} = \frac{b^4}{12}$	$\frac{b}{2}$
3. Triangle 	$\frac{bh}{2}$	$I_{xx} = \frac{bh^3}{36}$	$\frac{h}{3}$

4. Hollow rectangle 	$b(h - h_1)$	$I_{xx} = \frac{b}{12}(h^3 - h_1^3)$	$\frac{h}{2}$
5. Hollow square 	$b^2 - h^2$	$I_{xx} = I_{yy} = \frac{b^4 - h^4}{12}$	$\frac{b}{2}$
6. Trapezoidal 	$\frac{a+b}{2} \times h$	$I_{xx} = \frac{h^2(a^2 + 4ab + b^2)}{36(a+b)}$	$\frac{a+2b}{3(a+b)} \times h$

Section	(A)	(I)	(y)
7. Circle 	$\frac{\pi}{4} \times d^2$	$I_{xx} = I_{yy} = \frac{\pi d^4}{64}$	$\frac{d}{2}$
8. Hollow circle 	$\frac{\pi}{4}(d^2 - d_1^2)$	$I_{xx} = I_{yy} = \frac{\pi}{64}(d^4 - d_1^4)$	$\frac{d}{2}$
9. Elliptical 	πab	$I_{xx} = \frac{\pi}{4} \times a^3 b$ $I_{yy} = \frac{\pi}{4} \times ab^3$	a b

MOMENT OF INERTIA:

INTRODUCTION:

Moment of a force (P) about a point, is the product of the force and perpendicular distance (x) between the point and the line of action of the force (i.e. P.x). If this moment is again multiplied by the perpendicular distance (x) between the point and the line of action of the force i.e. P.x(x) = Px2, then this quantity is called moment of inertia. .

CALCULATION OF MOMENT OF INERTIABY INTEGRATION METHOD:

The moment of inertia of an area may be found out by the method of integration: Consider a plane figure, whose moment of inertia is required to be found out about X-X axis and Y-Y axis as shown in Fig 4.12. Let us divide the whole area into a no. of strips. Consider one of these strips.

Let dA= Area of the strip

x = Distance of the centre of gravity of the strip on X-X axis and

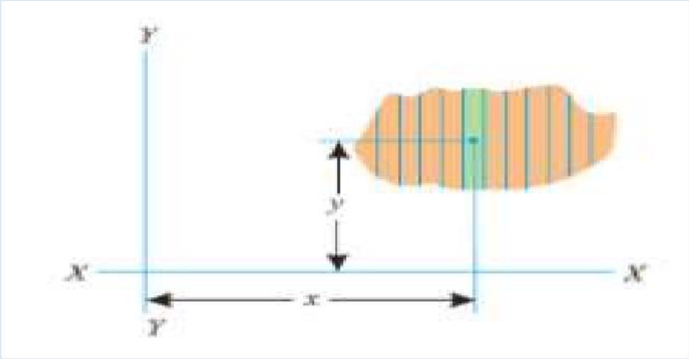
y = Distance of the centre of gravity of the strip on Y-Y axis.

We know that the moment of inertia of the strip about Y-Y axis = dA.x2

Now the moment of inertia of the whole area may be found out by integrating above equation. i.e.,

$$I_{YY} = \Sigma dA.x2$$

$$\text{Similarly } I_{XX} = \Sigma dA \,.\, y2$$



Unit: It depends on units of area and length If area=m2, length =m then,

M.I=m4 If area=mm2, length=mm then, M.I=mm4

THEOREM OF PERPENDICULAR AXIS

If IXX and IYY be the moments of inertia of a plane section about two perpendicular axis meeting at O, the moment of inertia IZZ about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:

$I_{ZZ}= I_{XX}+I_{YY}$

THEOREM OF PARALLEL AXIS

It states, If the moment of inertia of a plane area about an axis through its centre of gravity is denoted by IG, then moment of inertia of the area about any other axis AB, parallel to the first, and at a distance h from the centre of gravity is given by:

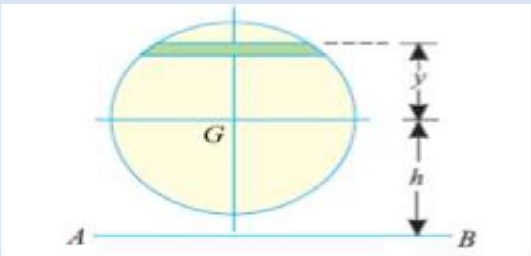
I_{AB}= IG+ ah2 Where

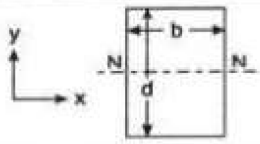
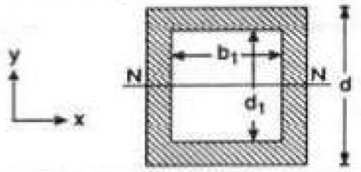
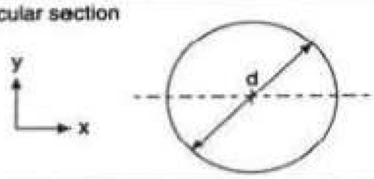
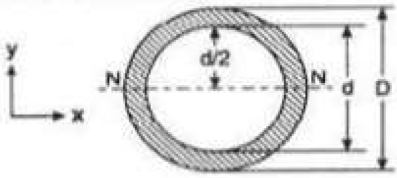
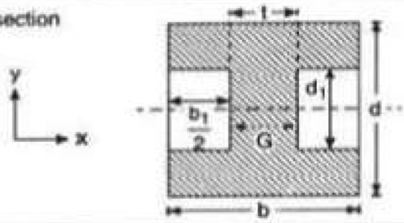
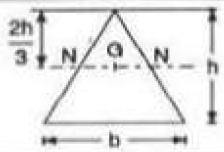
I_{AB}= Moment of inertia of the area about an axis AB,

I_G= Moment of Inertia of the area about its centre of gravity

a = Area of the section, and

h = Distance between centre of gravity of the section and axis AB

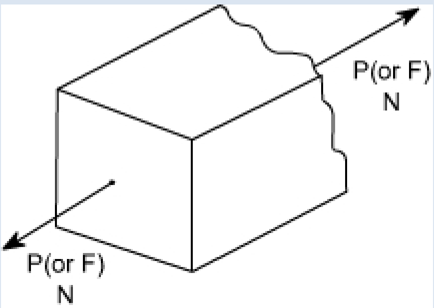


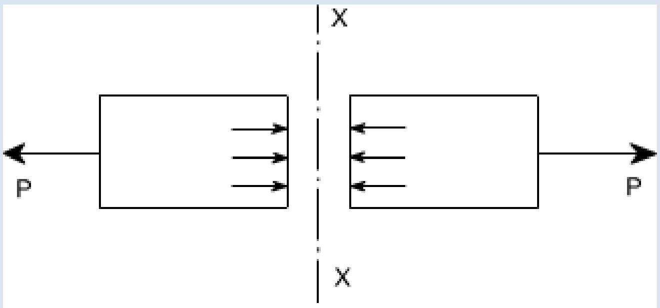
Type of section	Moment of Inertia	y_{max}	Section modulas (Z)
Rectangle or paralleogram 	$I_{xx} = \frac{bd^3}{12}$ $I_{yy} = \frac{db^3}{12}$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{bd^2}{6}$ $Z_{yy} = \frac{db^2}{6}$
Hollow rectangular section 	$I_{xx} = \frac{bd^3}{12} - \frac{b_1d_1^3}{12}$ $I_{yy} = \frac{db^3}{12} - \frac{d_1b_1^3}{12}$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{1}{6d}(bd^3 - b_1d_1^3)$ $Z_{yy} = \frac{1}{6b}(db^3 - d_1b_1^3)$
Circular section 	$I_{xx} = \frac{p}{64} d^4$ $I_{yy} = \frac{p}{64} d^4$	$\frac{d}{2}$ $\frac{d}{2}$	$Z_{xx} = \frac{p}{32} d^3$ $Z_{yy} = \frac{p}{32} d^3$
Hollow curcular section 	$I_{xx} = I_{yy} = I$ $I_{yy} = \frac{p}{64}(D^4 - d^4)$	$\frac{D}{2}$	$Z_{xx} = Z_{yy} = Z$ $Z = \frac{p}{32D}(D^4 - d^4)$
I-section 	$I_{xx} = \frac{bd^3}{12} - \frac{b_1d_1^3}{12}$ $I_{yy} = \frac{db^3}{12} - \frac{d_1b_1^3}{12}$ or $I_{xx} = \frac{1}{12}(hd^3 - (b - t)d_1^3)$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{1}{6d}(bd^3 - b_1d_1^3)$ $Z_{yy} = \frac{1}{6b}(db^3 - d_1b_1^3)$
Triangle 	$I_G = \frac{bh^3}{36}$	$\frac{2}{3}h$	$Z_G = \frac{bh^2}{24}$

CHAPTER-2:- SIMPLE & COMPLEX STRESS ,STRAIN

Stress

Stress is the internal resistance offered by the body to the external load applied to it per unit cross sectional area. Stresses are normal to the plane to which they act and are tensile or compressive in nature.





As we know that in mechanics of deformable solids, externally applied forces acts ona body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particlesof material due to cohesion. These internal forces give rise to a concept of stress. Consider a rectangular rod subjected to axial pull P. Let us imagine that the same rectangular bar is assumed to be cut into two halves at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown.

Now stress is defined as the force intensity or force per unit area. Here we use a symbol □ to represent the stress.

$$\square \; \square \; \frac{P}{A}$$

Where A is the area of the $X - X$ section

Here we are using an assumption that the total force or total load carried by the rectangular bar is uniformly distributed over its cross — section. But the stress distributionsmay be for from uniform, with local regions of high stress known as stress concentrations. If the force carried by a component is not uniformly distributed over its cross — sectional area, A, we must consider a small area, ‘δA’ which carries a small load ‘δP’, of the total force ‘P', Then definition of stress is

$$\boxed{\sigma = \frac{\delta F}{\delta A}}$$

As a particular stress generally holds true only at a point, therefore it is definedmathematically as

$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}}$$

Units :

The basic units of stress in S.I units i.e. (International system) are N / m^2 (or Pa) MPa =10^6 Pa

GPa = 10^9 Pa KPa

= 10^3 Pa

Sometimes N / mm^2 units are also used, because this is an equivalent to MPa.

TYPES OF STRESSES :

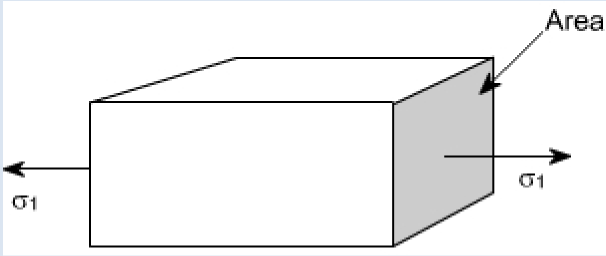
Only two basic stresses exists :

(1) normal stress

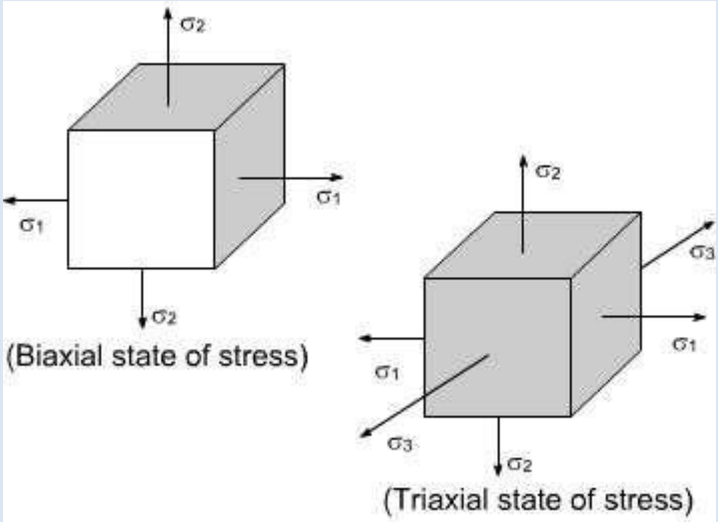
(2) shear stress.

Other stresses either are similar to these basic stresses or are a combination of this e.g. bending stress is a combination tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress. Let us define the normal stresses and shear stresses in the following sections.

Normal stresses : We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally denoted by a Greek letter (σ)

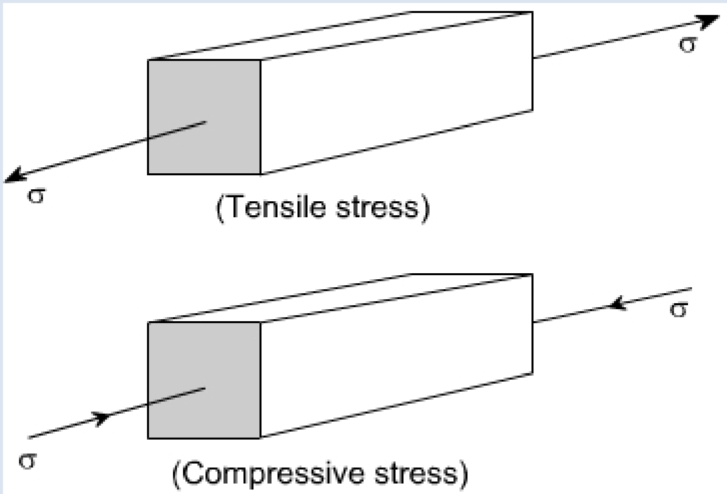


This is also known as uniaxial state of stress, because the stresses acts only in one direction however, such a state rarely exists, therefore we have biaxial and triaxial state of stresses where either the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses acts as shown in the figures below :

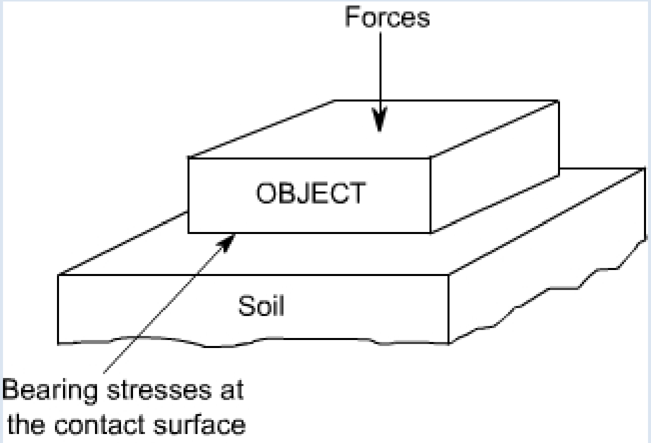


Tensile or compressive Stresses:

The normal stresses can be either tensile or compressive whether the stresses acts outof the area or into the area



Bearing Stress: When one object presses against another, it is referred to a bearingstress (They are in fact the compressive stresses).



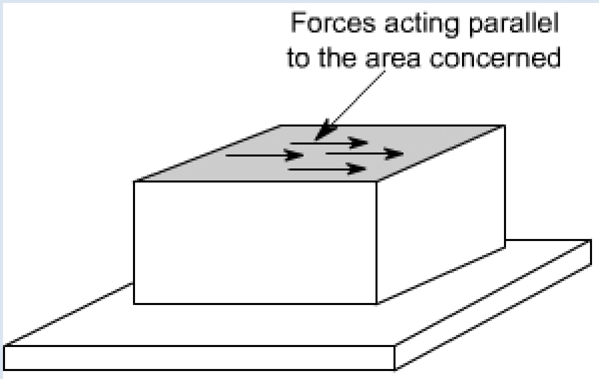
<u>Sign convections for Normal stress</u>

Direct stresses or normal stresses

- tensile +ve
- compressive –ve

<u>Shear Stresses:</u>

Let us consider now the situation, where the cross — sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces. The resulting stress is known as shear stress.



The resulting force intensities are known as shear stresses, the mean shear stress being equal to

$$\tau = \frac{P}{A}$$

Where P is the total force and A the area over which it acts. As we know that the particular stress generally holds good only at a point therefore we can define shear stress at a point as

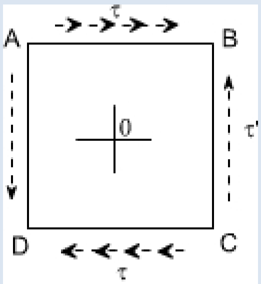
$$\tau = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

The Greek symbol τ (tau, suggesting tangential) is used to denote shear stress.

Complementary shear stresses:

The existence of shear stresses on any two sides of the element induces complementary shear stresses on the other two sides of the element to maintain equilibrium. As shown in the figure the shear stress τ in sides AB and CD induces a

complimentary shear stress τ' in sides AD and BC.

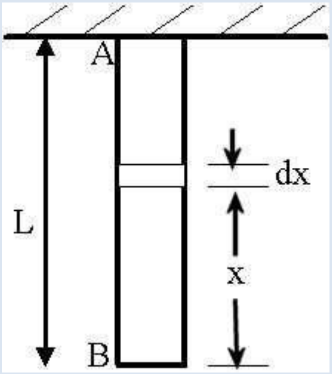


Sign convections for shear stresses:

- tending to turn the element C.W +ve.
- tending to turn the element C.C.W – ve.

Deformation of a Body due to Self Weight

Consider a bar AB hanging freely under its own weight as shown in the figure.



Let

L= length of the bar

A= cross-sectional area of the bar

E= Young's modulus of the bar material w=

specific weight of the bar material

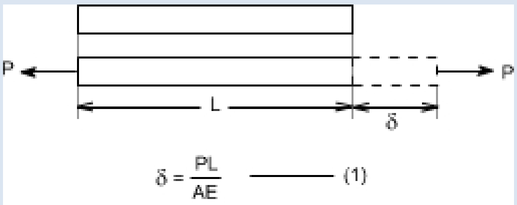
Then deformation due to the self-weight of the bar is

$$\boxed{\delta L = \frac{WL}{2E}}$$

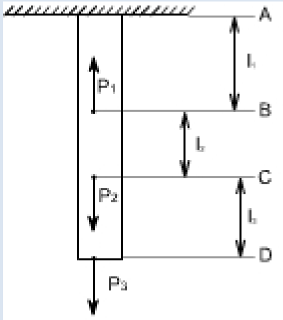
Members in Uni – axial state of stress

Introduction: [For members subjected to uniaxial state of stress]

For a prismatic bar loaded in tension by an axial force P, the elongation of the bar canbe determined as



Suppose the bar is loaded at one or more intermediate positions, then equation (1) can be readily adapted to handle this situation, i.e. we can determine the axial force in each part of the bar i.e. parts AB, BC, CD, and calculate the elongation or shortening of each part separately, finally, these changes in lengths can be added algebraically toobtain the total charge in length of the entire bar.



When either the axial force or the cross — sectional area varies continuosly along the axis of the bar, then equation (1) is no longer suitable. Instead, the elongation can be found by considering a deferential element of a bar and then the equation (1) becomes

$$d\delta = \frac{P_x\,dx}{E.A_x}$$

$$\delta = \int_0^l \frac{P_x\,dx}{E.A_x}$$

i.e. the axial force $P_x$and area of the cross – section A_x must be expressed as functions of x. If the expressions for $P_x$and A_x are not too complicated, the integral can be evaluated analytically, otherwise Numerical methods or techniques can be used to evaluate these integrals.

Principle of Superposition

The principle of superposition states that when there are numbers of loads are acting together on an elastic material, the resultant strain will be the sum of individual strains.

Strain:

When a single force or a system force acts on a body, it undergoes some deformation. This deformation per unit length is known as strain. Mathematically strain may be defined as deformation per unit length.

So,

Strain=Elongation/Original length

Or, $\varepsilon = \frac{\delta l}{l}$

Elasticity:

The property of material by virtue of which it returns to its original shape and size upon removal of load is known as elasticity.

Hooks Law

It states that within elastic limit stress is proportional to strain. Mathematically $E = \frac{Stress}{Strain}$

Where E = Young's Modulus

Hooks law holds good equally for tension and compression.

Poisson's Ratio:

The ratio lateral strain to longitudinal strain produced by a single stress is known as Poisson's ratio. Symbol used for poisson's ratio is μ or $1/m$.

Modulus of Elasticity (or Young's Modulus)

Young's modulus is defined as the ratio of stress to strain within elastic limit.

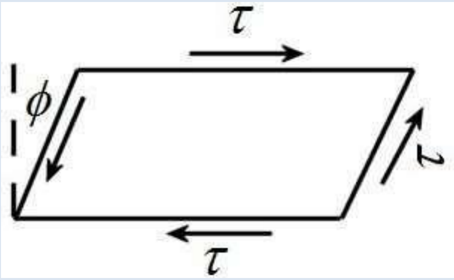
Deformation of a body due to load acting on it

We know that young's modulus $E = \frac{\text{Stress}}{\text{Strain}}$

$$\boxed{\delta l = \frac{Pl}{AE}}$$

Shear Strain

The distortion produced by shear stress on an element or rectangular block is shown inthe figure. The shear strain or 'slide' is expressed by angle ϕ and it can be defined as the change in the right angle. It is measured in radians and is dimensionless in nature.



Modulus of Rigidity

For elastic materials it is found that shear stress is proportional to the shear strain within elastic limit. The ratio is called modulus rigidity. It is denoted by the symbol 'G' or 'C'.

$$G = \frac{\text{shear stress}}{\text{shear strain}} \quad N/mm^2$$

Bulk modulus (K): It is defined as the ratio of uniform stress intensity to the volumetric strain. It is denoted by the symbol K.

$$K = \frac{\text{stress intensity}}{\text{volumetric strain}}$$

Relation between elastic constants:

Elastic constants: These are the relations which determine the deformations producedby a given stress system acting on a particular material. These factors are constant within elastic limit, and known as modulus of elasticity *E*, modulus of rigidity *G*, Bulk modulus *K* and Poisson's ratio *μ*.

Relationship between modulus of elasticity (E) and bulk modulus (K):

$$\boxed{E = 3K(1-2\mu)}$$

Relationship between modulus of elasticity (E) and modulus of rigidity (G):

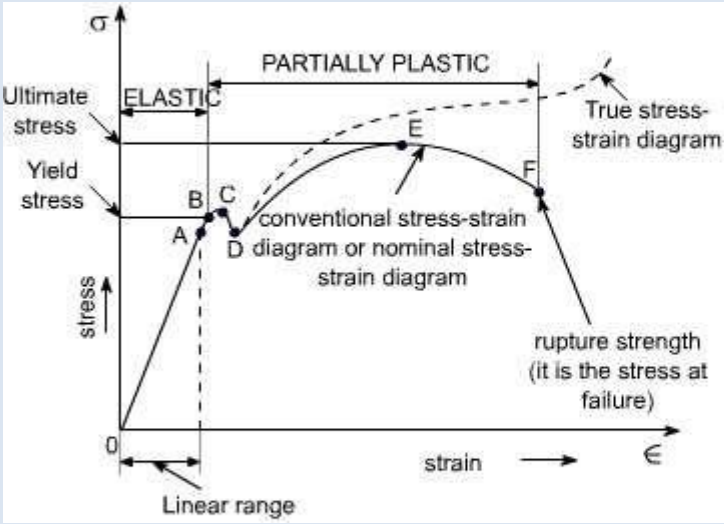
$$\boxed{E = 2G(1+\mu)}$$

Relation among three elastic constant

$$\boxed{E = \frac{9KG}{G+3K}}$$

Stress – strain diagram for mild steel

A typical tensile test curve for the mild steel has been shown below



SALIENT POINTS OF THE GRAPH:

(A) So it is evident form the graph that the strain is proportional to strain or elongation is proportional to the load giving a st.line relationship. This law of proportionality is valid upto a point A.

or we can say that point A is some ultimate point when the linear nature of the graph ceases or there is a deviation from the linear nature. This point is known as **the limit of proportionality or the proportionality limit**.

(B) For a short period beyond the point A, the material may still be elastic in the sense that the deformations are completely recovered when the load is removed. The limiting point B is termed as **Elastic Limit** .

(C) and (D) - Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. There will be thus permanent deformation or permanent set

(E) A further increase in the load will cause marked deformation in the whole volume of the metal. The maximum load which the specimen can with stand without failure is called the load at the ultimate strength.

The highest point ‘E' of the diagram corresponds to the ultimate strength of a material.

s_u = Stress which the specimen can with stand without failure & is known as Ultimate Strength or Tensile Strength.

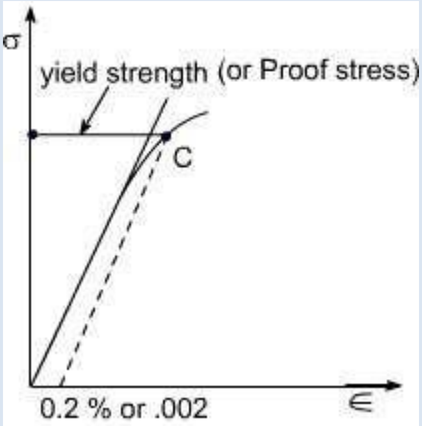
s_u is equal to load at E divided by the original cross-sectional area of the bar.

(F) Beyond point E, the bar begins to forms neck. The load falling from the maximum until fracture occurs at F. Beyond point E, the cross-sectional area of the specimen begins to reduce rapidly over a relatively small length of bar and the bar is said to form a neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point F.

when load is removed. These two points are termed as upper and lower yield points respectively. The stress at the yield point is called the yield strength.

A study a stress — strain diagrams shows that the yield point is so near the proportional limit that for most purpose the two may be taken as one. However, it is much easier to locate the former. For material which do not posses a well define yield points, In order to find the yield point or yield strength, an offset method is applied.

In this method a line is drawn parallel to the straight line portion of initial stress diagramby off setting this by an amount equal to 0.2% of the strain as shown as below and this happens especially for the low carbon steel.



Ductile and Brittle Materials:

Based on this behaviour, the materials may be classified as ductile or brittlematerials

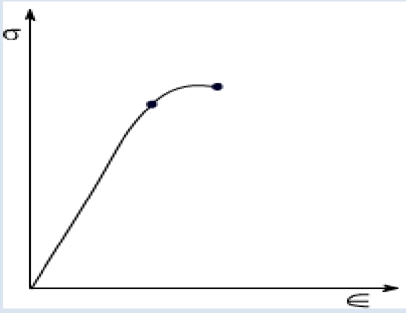
Ductile Materials:

It we just examine the earlier tension curve one can notice that the extension of the materials over the plastic range is considerably in excess of that associated with elastic loading. The Capacity of materials to allow these large deformations or large extensions without failure is termed as ductility. The materials with high ductility are termed as ductile materials.

Brittle Materials:

A brittle material is one which exhibits a relatively small extensions or deformations to fracture, so that the partially plastic region of the tensile test graph is much reduced.

This type of graph is shown by the cast iron or steels with high carbon contents or concrete



Mechanical Properties of material:

Elasticity: Property of material by virtue of which it can regain its shape after removalof external load

Plasticity: Property of material by virtue of which, it will be in a state of permanent deformation even after removal of external load.

Ductility: Property of material by virtue of which, the material can be drawn intowires.

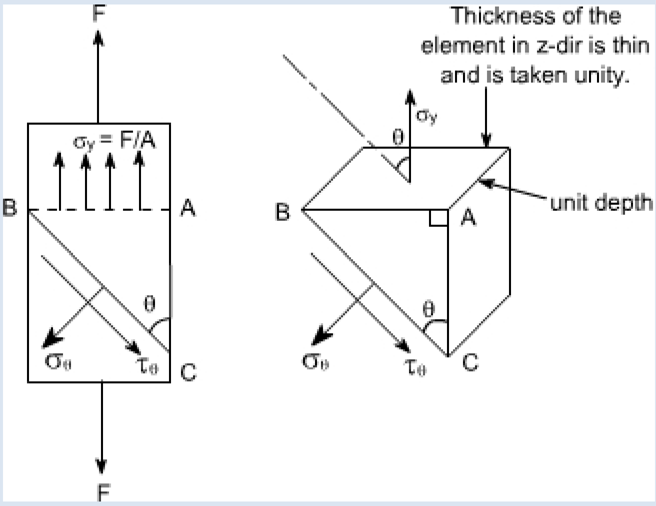
Hardness: Property of material by virtue of which the material will offer resistance to penetration or indentation.

Stresses on oblique plane:

Till now we have dealt with either pure normal direct stress or pure shear stress. In many instances, however both direct and shear stresses acts and the resultant stress across any section will be neither normal nor tangential to the plane. A plane stse of stress is a 2 dimensional stae of stress in a sense that the stress components in one direction are all zero i.e

$\sigma_z = \tau_{yz} = \tau_{zx} = 0$

Examples of plane state of stress include plates and shells. Consider the general case of a bar under direct load F giving rise to a stress σ_y vertically



The stress acting at a point is represented by the stresses acting on the faces of the element enclosing the point. The stresses change with the inclination of the planes passing through that point i.e. the stress on the faces of the element vary as the angular position of the element changes. Let the block be of unit depth now considering the equilibrium of forces on the triangle portion ABC. Resolving forces perpendicular to BC, gives

$\sigma_\theta .BC.1 = \sigma_y \sin\theta \ . AB.1$

but $AB/BC = \sin\theta$ or $AB = BC \sin\theta$

Substituting this value in the above equation, we get

$\sigma_\theta .BC.1 = \sigma_y \sin\theta \ . BC \sin\theta \ . 1$ or $\sigma_\theta = \sigma_y \sin^2 2\theta$

Now resolving the forces parallel to BC

$\tau_\theta .BC.1 = \sigma_y \cos\theta \ . AB \sin. 1$

again $AB = BC \cos\theta$

(1)

□ $_{□}$ .BC.1 = □ $_y$ cos □ . BC sin □ .1 or □ $_{□}$ =□ $_y$ sin□
cos□

$$□ \quad □ \quad \frac{1}{2} .□_{□\, y} \quad \text{sin}\, 2□ \qquad (2)$$

If □ = 90^0 the BC will be parallel to AB and □$_{□}$ = 0, i.e. there will be only direct stress or normal stress.

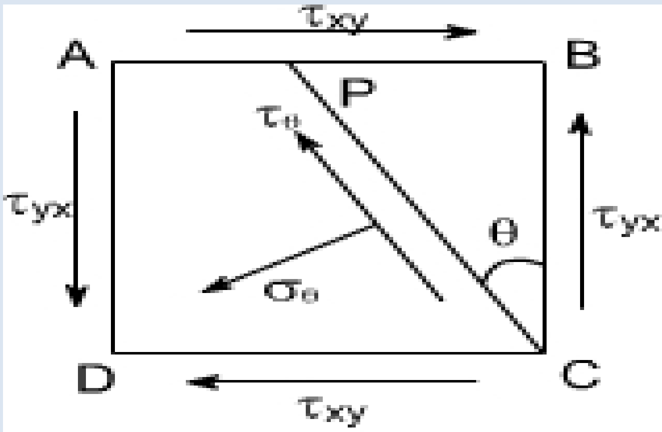
By examining the equations (1) and (2), the following conclusions may be drawnThe value

of direct stress □$_{□}$ is maximum and is equal to □ $_y$ when v= 90^0.

The shear stress □$_{□}$ has a maximum value of 0.5 □ $_y$ when □ = 45^0

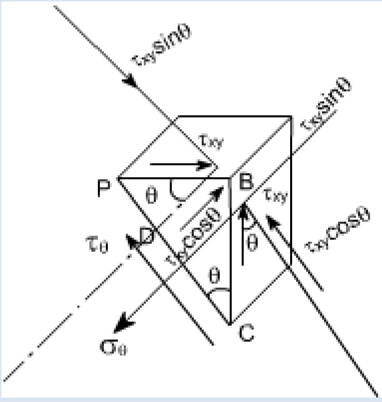
Material subjected to pure shear:

Consider the element shown to which shear stresses have been applied to the sides ABand DC



Complementary shear stresses of equal value but of opposite effect are then set up onthe sides AD and BC in order to prevent the rotation of the element. Since the applied and complementary shear stresses are of equal value on the x and y planes. Therefore, they are both represented by the symbol □ $_{xy}$.

Now consider the equilibrium of portion of PBC



Assuming unit depth and resolving normal to PC or in the direction of σ_θ

$\sigma_\theta .PC.1 = \tau_{xy} .PB.\cos\theta \;.1 + \tau_{xy} .BC.\sin\theta \;.1$

$= \tau_{xy} .PB.\cos\theta + \tau_{xy} .BC.\sin\theta$

Now writing PB and BC in terms of PC so that it cancels out from the two sides PB/PC = $\sin\theta$
BC/PC = $\cos\theta$

$\sigma_\theta .PC.1 = \tau_{xy} .\cos\theta \; \sin\theta \; PC + \tau_{xy} .\cos\theta \; .\sin\theta \; .PC$

$\sigma_\theta = 2\tau_{xy} \sin\theta \; \cos\theta$

Or, $\sigma_\theta = 2\tau_{xy} \sin 2\theta$ (1)

Now resolving forces parallel to PC or in the direction of τ_θ .then τ_{xy} PC.1

$= \tau_{xy} . PB \sin\theta - \tau_{xy} BC \cos\theta$

-ve sign has been put because this component is in the same direction as that of τ_θ .again converting the various quantities in terms of PC we have

$\tau_\theta PC.1 = \tau_{xy} . PB.\sin^2\theta - \tau_{xy} PC\cos^2\theta$

$= -\tau_{xy} [\cos^2\theta - \sin^2\theta]$

$= -\tau_{xy} \cos 2\theta$ (2)

the negative sign means that the sense of τ_θ is opposite to that of assumed one. Let us examine the equations (1) and (2) respectively

From equation (1) i.e,

$\sigma_\theta = \tau_{xy} \sin 2\theta$

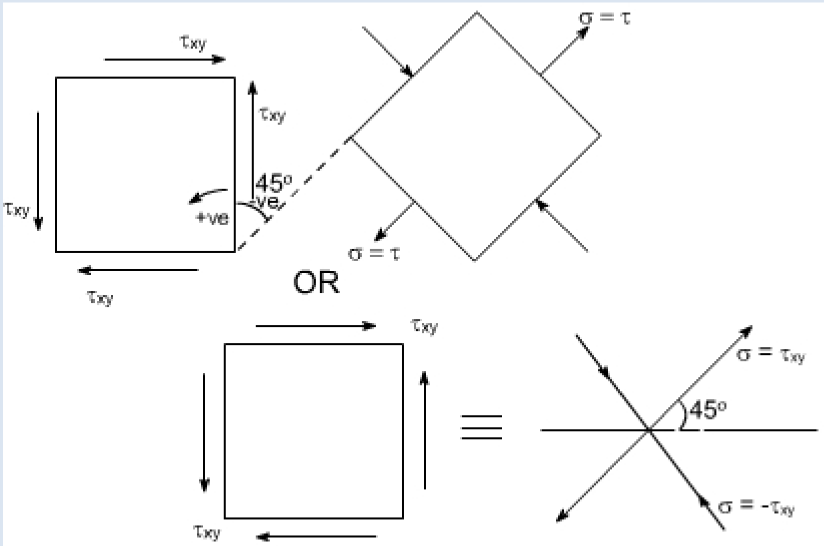
The equation (1) represents that the maximum value of σ_θ is τ_{xy} when $\theta = 45^0$.Let us take into consideration the equation (2) which states that

$\tau_{\theta} = -\tau_{xy} \cos 2\theta$

It indicates that the maximum value of τ_{θ} is τ_{xy} when $\theta = 0^0$ or 90^0. it has a value zero when $\theta = 45^0$.

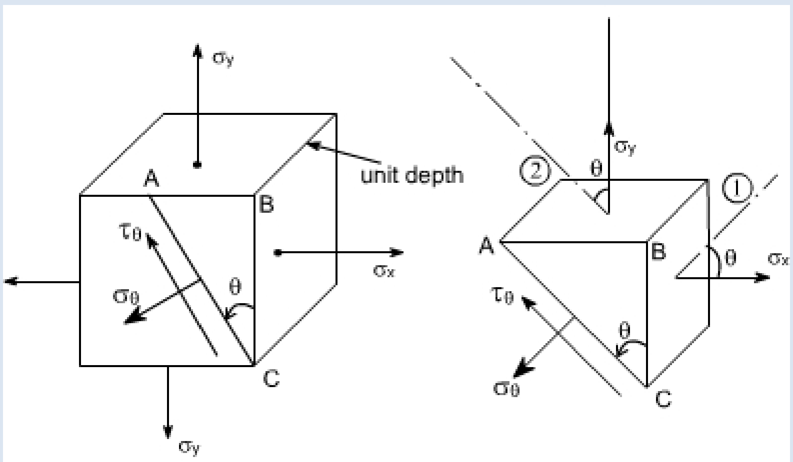
From equation (1) it may be noticed that the normal component has maximum and minimum values of $+\tau_{xy}$ (tension) and τ_{xy}(compression) on plane at $\pm 45^0$ to the applied shear and on these planes the tangential component is zero.

Hence the system of pure shear stresses produces and equivalent direct stress system, one set compressive and one tensile each located at 45^0 to the original shear directionsas depicted in the figure below:



<u>Material subjected to two mutually perpendicular direct stresses:</u>

Now consider a rectangular element of unit depth, subjected to a system of two direct stresses both tensile, σ_x and σ_y acting right angles to each other.



for equilibrium of the portion ABC, resolving perpendicular to AC

□$_{□}$. AC.1 = □$_y$ sin □ . AB.1 + □$_x$ cos □ . BC.1

converting AB and BC in terms of AC so that AC cancels out from the sides

□$_{□}$ = □ sin^2_y□ +□ cos^2_x□

Futher, recalling that cos^2□ - sin^2□ = cos2□ or (1 - cos2□)/2 = sin^2□ Similarly(1 + cos2□)/2 = cos^{2}q

Hence by these transformations the expression for reduces to

= 1/2 σ_y (1+ cos2Φ) + 1/2 σ_x (1 + cos2□□

) On rearranging the various terms we get

$$\boxed{\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta} \qquad (3) \qquad (3)$$

The — ve sign appears because this component is in the same direction as that of AC. Again converting the various quantities in terms of AC so that the AC cancels out fromthe two sides.

$$\tau_\theta . AC.1 = [\tau_x \cos\theta \sin\theta - \sigma_y \sin\theta \cos\theta] AC$$

$$\tau_\theta = (\sigma_x - \sigma_y)\sin\theta\cos\theta$$

$$= \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

$$\text{or} \quad \boxed{\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta} \qquad (4) \qquad (4)$$

Conclusions :

The following conclusions may be drawn from equation (3) and (4)

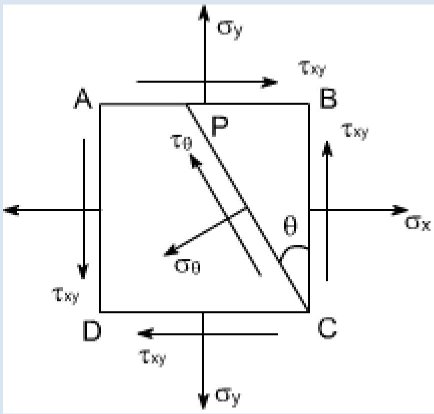
1. The maximum direct stress would be equal to $_x$ or $_y$ whichever is the greater, when = 0^0 or 90^0

2. The maximum shear stress in the plane of the applied stresses occurs when = 45^0

$$\tau_{max} = \frac{(\sigma_x - \sigma_y)}{2}$$

<u>Material subjected to combined direct and shear stresses</u>:

Now consider a complex stress system shown below, acting on an element of material. The stresses $_{x}$ and $_{y}$ may be compressive or tensile and may be the result of directforces or as a result of bending.The shear stresses may be as shown or completelyreversed and occur as a result of either shear force or torsion as shown in the figurebelow:



As per the double subscript notation the shear stress on the face BC should be notifiedas $_{yx}$, however, we have already seen that for a pair of shear stresses there is a set of complementary shear stresses generated such that

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material subjected to pure shear. In this case the various formulas deserved are as follows

(ii) Material subjected to two mutually perpendicular direct stresses. In this case the various formula's derived are as follows.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

To get the required equations for the case under consideration,let us add the respective equations for the above two cases such that

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

These are the equilibrium equations for stresses at a point. They do not depend onmaterial proportions and are equally valid for elastic and inelastic behaviour

This eqn gives two values of 2 that differ by 180^0 .Hence the planes on which maximumand minimum normal stresses occurate 90^0apart.

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

$$= 0$$

$$\text{i.e.} -(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$$

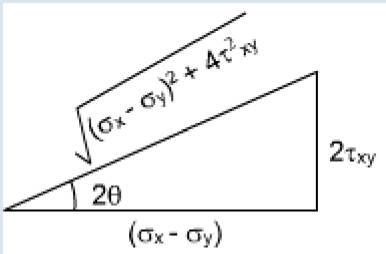
$$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$$

Thus, $$\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$$

From the triangle it may be determined

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$



Substituting the values of cos2 and sin2 in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2} \cdot \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

$$+ \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2} \cdot \frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

$$+ \frac{1}{2}\frac{4\tau^2{}_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2} \cdot \frac{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2} \cdot \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}} \cdot \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2} \cdot \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2} \cdot \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2} \cdot \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}} - \frac{\tau_{xy}.(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}}$$

$$\tau_\theta = 0$$

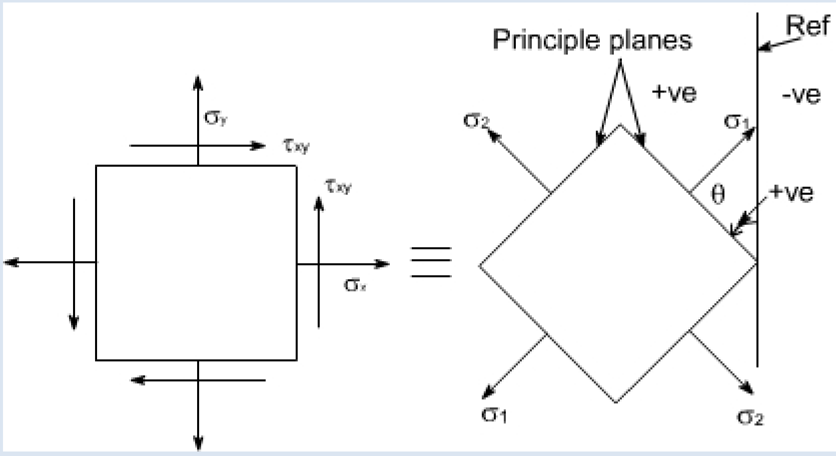
This shows that the values oshear stress is zero on the principal planes.

Hence the maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

will yield two values of 2 separated by 180^0 i.e. two values of separated by 90^0 .Thus the two principal stresses occur on mutually perpendicular planes termed principal planes.

Therefore the two — dimensional complex stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a material subjected to direct stresses the value of maximum shear stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y)$ at $\theta = 45^0$,Thus, for a 2-dimensional state of stress,subjected to principle stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$, on substituting the values if σ_1 and σ_2,we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$

$$= 0$$

or $(\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p . \tan 2\theta_s = 1}$$

Therefore,it can be concluded that the equation (2) is a negative reciprocal of equation (1)hence the roots for the double angle of equation (2) are 90^0 away from the corresponding angle of equation (1).

This means that the angles that angles that locate the plane of maximum or minimum shearing stresses form angles of 45^0 with the planes of principal stresses.

Futher, by making the triangle we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2{}_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2{}_{xy}}}$$

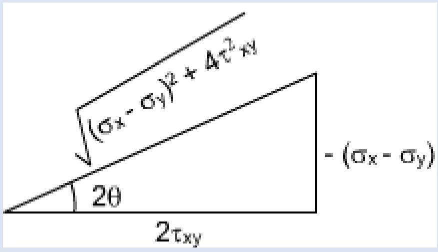
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}.-\frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2{}_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2{}_{xy}}}$$

$$= -\frac{1}{2}.\frac{(\sigma_y - \sigma_x)^2 + 4\tau^2{}_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2{}_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}$$



Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.

The largest stress regard less of sign is always know as maximum shear stress.

Principal plane inclination in terms of associated principal stress:

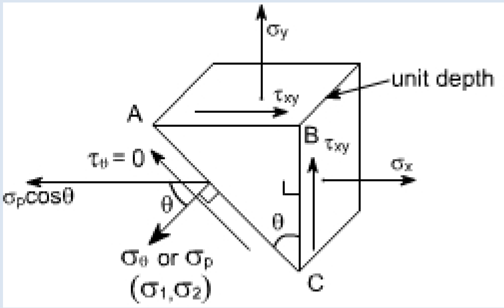
We know that the equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

yields two values of q i.e. the inclination of the two principal planes on which the principal stresses s_1 and s_2 act. It is uncertain,however, which stress acts on which plane unless equation.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$ is used and observing which one of the two principal stresses is obtained.

Alternatively we can also find the answer to this problem in the following manner



Consider once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses $_p$ acts, and the shear stress is zero.

Resolving the forces horizontally we get:

$_x$.BC . 1 + $_{xy}$ .AB . 1 = $_p$. cos . AC dividing the above equation through by BC we get

$$\sigma_x . + \tau_{xy}\frac{AB}{BC} = \sigma_p . \cos\theta . \frac{AC}{BC}$$

or

$$\sigma_x + \tau_{xy}\tan\theta = \sigma_p$$

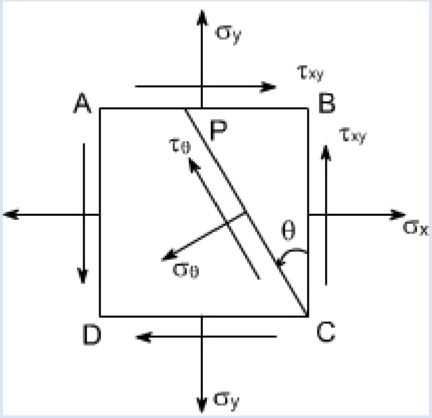
Thus

$$\boxed{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}}$$

GRAPHICAL SOLUTION – MOHR'S STRESS CIRCLE

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This grapical representation is very useful in depending the relationships between normal and shear stresses acting on any inclined plane at a pointin a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure



The above system represents a complete stress system for any condition of applied load in two dimensions

The Mohr's stress circle is used to find out graphically the direct stress and sheer stress on any plane inclined at to the plane on which $_x$ acts.The direction of here is takenin anticlockwise direction from the BC.

STEPS:

In order to do achieve the desired objective we proceed in the following manner

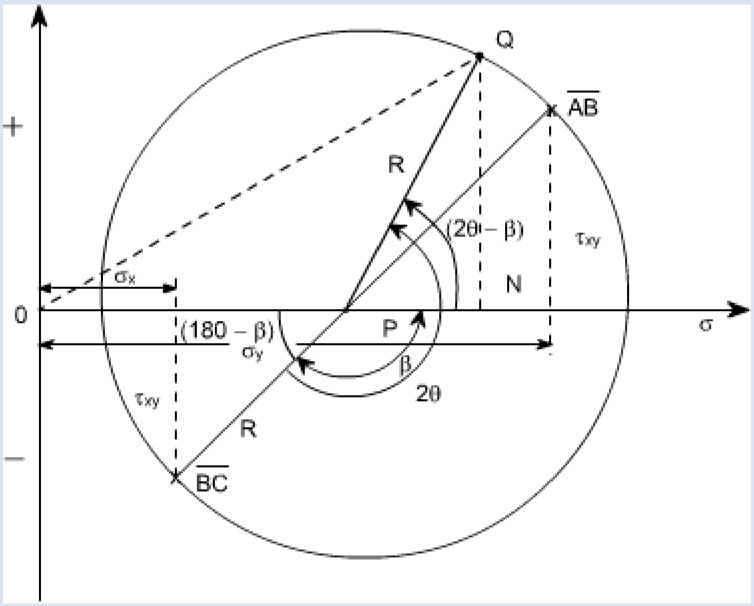
1. Label the Block ABCD.
2. Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)
3. Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.
4. Direct stresses tensile positive; compressive, negative Shear stresses
5. tending to turn block clockwise, positive. tending to turn block counter clockwise, negative

[i.e shearing stresses are +ve when its movement about the centre of the elementis clockwise]

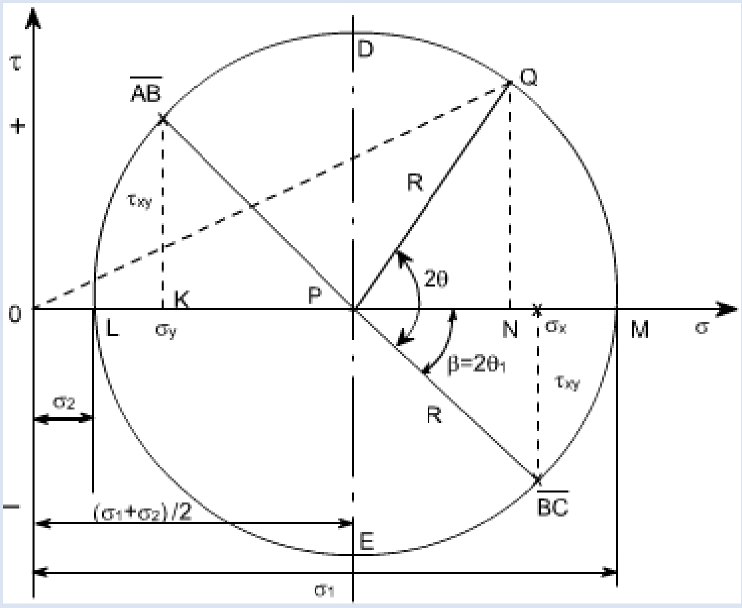
This gives two points on the graph which may than be labeled as respectively todenote stresses on these planes.

(i) join $\overline{AB}$ and $\overline{BC}$

(ii) The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining is diameter. Therefore the circle can now be drawn. Now every point on the circle then represents a state of stress on some plane through C.



Proof:



(1) The direct stress is maximum when Q is at M and at this point obviously thesheer stress is zero, hence by definition OM is the length representing the maximum principal stresses $_1$ and 2 $_1$ gives the angle of the plane $_1$ from BC. Similar OL is the other principal stress and is represented by $_2$

(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle.

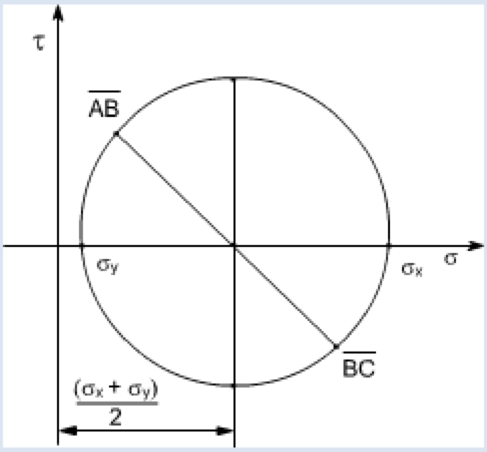
This follows that since shear stresses and complimentary sheer stresses have the same value; therefore the centre of the circle will always lie on the s axis midway between $_x$ and $_y$. [since + $_{xy}$ & $_{xy}$ are shear stress & complimentary shear stress so they are same in magnitude but different in sign.]

(3) From the above point the maximum sheer stress i.e. the Radius of the Mohr's stress circle would be

$$\frac{(\sigma_x - \sigma_y)}{2}$$

While the direct stress on the plane of maximum shear must be mid — may between $_x$ and $_y$ i.e

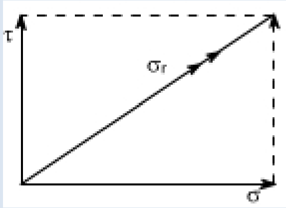
$$\frac{(\sigma_x + \sigma_y)}{2}$$



(4) As already defined the principal planes are the planes on which the shearcomponents are zero.

Therefore are conclude that on principal plane the sheer stress is zero.

(5) Since the resultant of two stress at 90^0 can be found from the parallogram of vectors as shown in the diagram.Thus, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.



(6) The graphical method of solution for a complex stress problems using Mohr's circle is a very powerful technique, since all the information relating to any plane within the stressed element is contained in the single construction. It thus, provides a convenient and rapid means of solution. Which is less prone to arithmetical errors and is highly recommended.

Numericals:

Let us discuss few representative problems dealing with complex state of stress to be solved either analytically or graphically.

Q 1: A circular bar 40 mm diameter carries an axial tensile load of 105 kN. What is the Value of shear stress on the planes on which the normal stress has a value of 50 MN/m^2 tensile.

Solution:

Tensile stress = F / A = 105 x 10^3 / x (0.02)^2

= 83.55 MN/m^2

Now the normal stress on an obliqe plane is given by the relation

= AB $_y$sin^2

50 x 10^6 = 83.55 MN/m^2 x 10^{6}sin^2

= 50^{0}68'

The shear stress on the oblique plane is then given by

= 1/2 AB $_y$sin2

= 1/2 x 83.55 x 10^6 x sin 101.36

= 40.96 MN/m^2

Therefore the required shear stress is 40.96 MN/m^2

Q2:

For a given loading conditions the state of stress in the wall of a cylinder is expressed as follows:

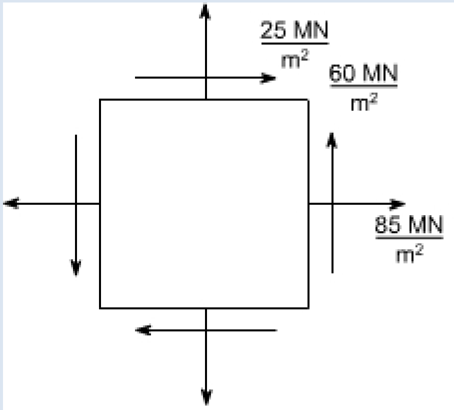
(a) 85 MN/m^2 tensile

(b) 25 MN/m^2 tensile at right angles to (a)

(c) Shear stresses of 60 MN/m^2 on the planes on which the stresses (a) and (b) act; the sheer couple acting on planes carrying the 25 MN/m^2 stress is clockwise in effect. Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged

Solution:

The problem may be attempted both analytically as well as graphically. Let us first obtainthe analytical solution



The principle stresses are given by the formula

$$\sigma_1 \text{ and } \sigma_2$$
$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$
$$= \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 + 25)^2 + (4 \times 60^2)}$$
$$= 55 \pm \frac{1}{2}.60\sqrt{5} = 55 \pm 67$$
$$\Rightarrow \sigma_1 = 122 \text{ MN/m}^2$$
$$\sigma_2 = -12 \text{ MN/m}^2 (\text{compressive})$$

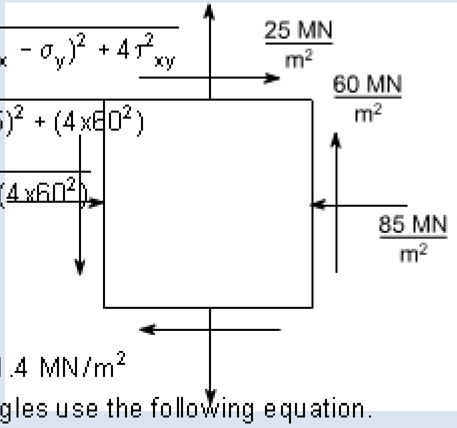
For finding out the planes on which the principle stresses act us the equation

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

The solution of this equation will yeild two values i.e they $_1$ and $_2$ giving $_1$= 31^{0}71' & $_2$= 121^{0}71'

(b) In this case only the loading (a) is changed i.e. its direction had been changed. Whilethe other stresses remains unchanged hence now the block diagram becomes.

Again the principal stresses would be given by the equation.



$$\sigma_1 \,\&\, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{1}{2}(-85 + 25) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= \frac{1}{2}(-60) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= -30 \pm \frac{1}{2}\sqrt{12100 + 14400}$$

$$= -30 \pm 81.4$$

$$\sigma_1 = 51.4\ \mathrm{MN/m^2};\ \sigma_2 = -111.4\ \mathrm{MN/m^2}$$

Again for finding out the angles use the following equation.

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

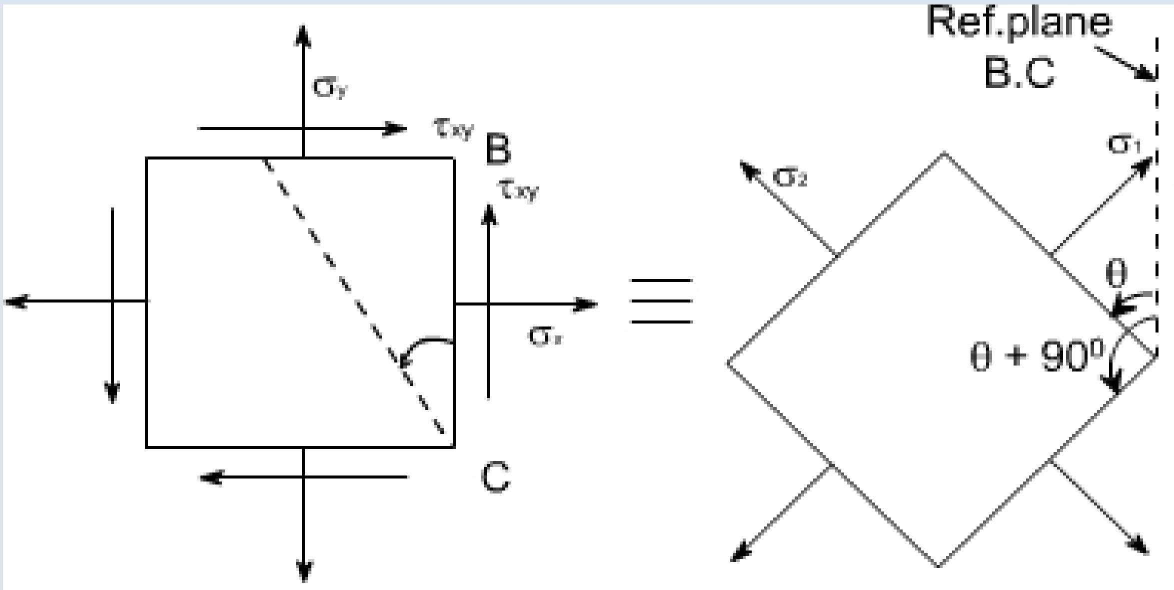
$$= \frac{2 \times 60}{-85 - 25} = +\frac{120}{-110}$$

$$= -\frac{12}{11}$$

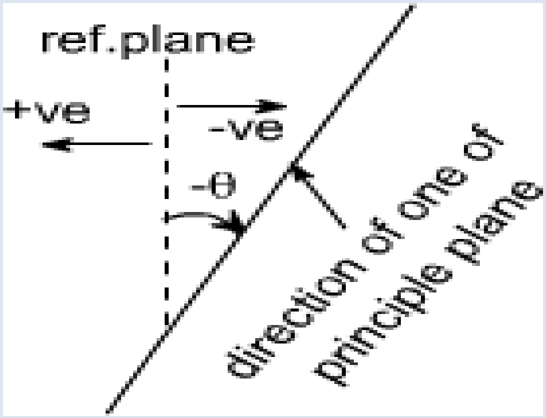
$$2\theta = \tan\left(-\frac{12}{11}\right)$$

$$\Rightarrow \quad \theta = -23.74^0$$

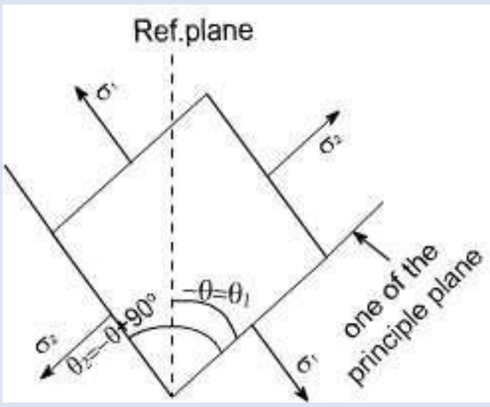
Thus, the two principle stresses acting on the two mutually perpendicular planes i.e principle planes may be depicted on the element as shown below:



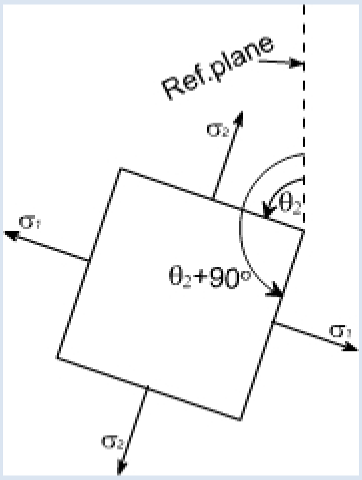
So this is the direction of one principle plane & the principle stresses acting on this would be $_1$ when is acting normal to this plane, now the direction of other principal plane would be 90^0 + because the principal planes are the two mutually perpendicular plane, hence rotate the another plane+ 90^0 in the same direction to get the another plane, now complete the material element if is negative that means we are measuring the angles in the opposite direction to the reference plane BC .



Therefore the direction of other principal planes would be { + 90} since the angle is always less in magnitude then 90 hence the quantity (+ 90) would be positive therefore the Inclination of other plane with reference plane would be positive therefore if just complete the Block. It would appear as



If we just want to measure the angles from the reference plane, than rotate this blockthrough 180^0 so as to have the following appearance.



So whenever one of the angles comes negative to get the positive value, firstAdd 90^0 to the value and again add 90^0 as in this case = $23^0 74'$

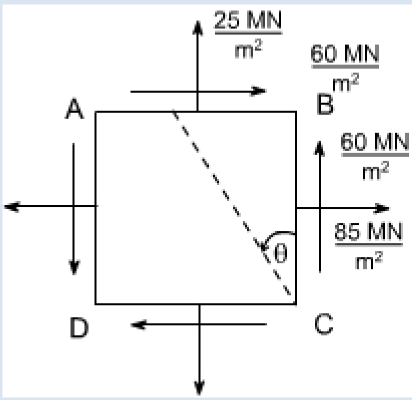
so $_1$ = $23^0 74'$ + 90^0 = $66^0 26'$ .Again adding 90^0 also gives the direction of other principle planes

i.e $_2$ = $66^0 26'$ + 90^0 = $156^0 26'$

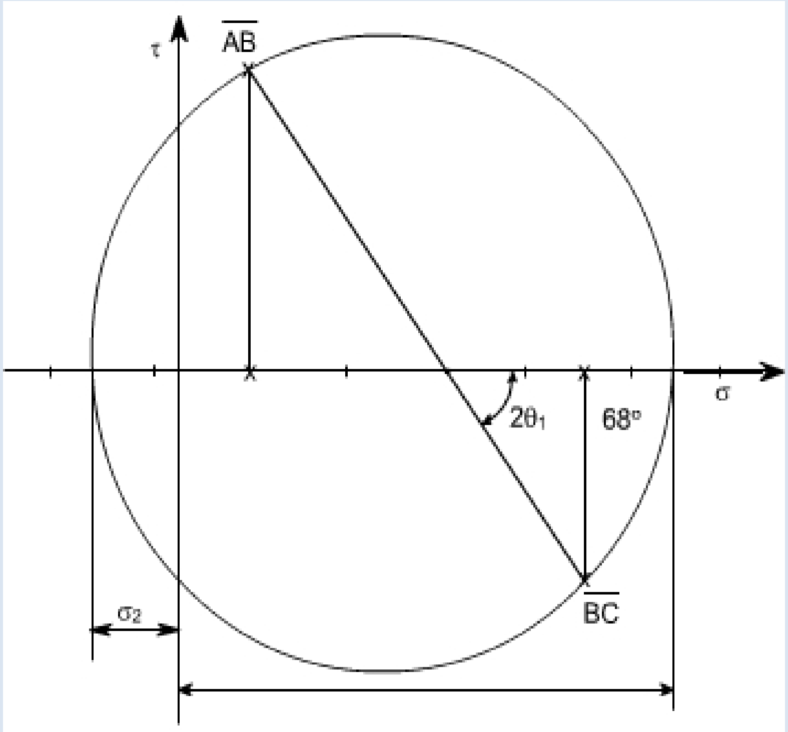
This is how we can show the angular position of these planes clearly.

GRAPHICAL SOLUTION:

Mohr's Circle solution: The same solution can be obtained using the graphical solution i.e the Mohr's stress circle,for the first part, the block diagram becomes



Construct the graphical construction as per the steps given earlier.



Taking the measurements from the Mohr's stress circle, the various quantities computedare

$_1 = 120$ MN/m^2 tensile

$_2 = 10$ MN/m^2 compressive

$_1 = 34^0$ counter clockwise from BC

$_2 = 34^0 + 90 = 124^0$ counter clockwise from BC

Part Second : The required configuration i.e the block diagram for this case is shownalong with the stress circle.

By taking the measurements, the various quantites computed are given as

$_1 = 56.5$ MN/m^2 tensile

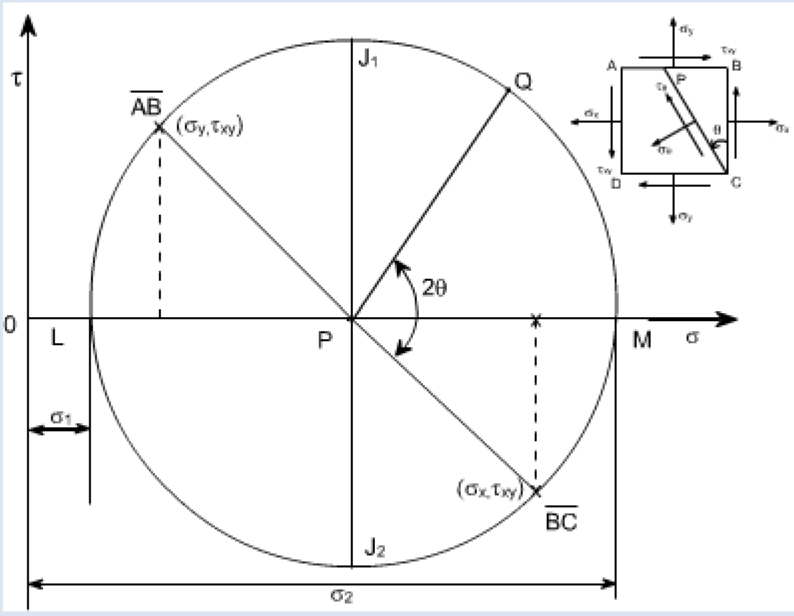
$_2$ = 106 MN/m^2 compressive

$_1$ = 66^{0}15' counter clockwise from BC

$_2$ = 156^{0}15' counter clockwise from BC

Salient points of Mohr's stress circle:

1. complementary shear stresses (on planes 90^0 apart on the circle) are equal in magnitude
2. The principal planes are orthogonal: points L and M are 180^0 apart on the circle (90^{0}apart in material)
3. There are no shear stresses on principal planes: point L and M lie on normalstress axis.
4. The planes of maximum shear are 45^0 from the principal points D and E are 90^0 , measured round the circle from points L and M.
5. The maximum shear stresses are equal in magnitude and given by points D and E 6.The normal stresses on the planes of maximum shear stress are equal i.e. points Dand E both have normal stress co-ordinate which is equal to the two principal stresses.



As we know that the circle represents all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co- ordinates of the point ‘Q' are seen to be the same as those derived from equilibrium of the element. i.e. the normal and shear stress components on any plane passing through the point can be found using Mohr's circle. Worthy of note:

1. The sides AB and BC of the element ABCD, which are 90^0 apart, are represented on the circle by $\overline{AB}P$ and $\overline{BC}P$ and they are 180^0 apart.

2. It has been shown that Mohr's circle represents all possible states at a point. Thus, it canbe seen at a point. Thus, it, can be seen that two planes LP and PM, 180^0 apart on the diagram and therefore 90^0 apart in the material, on which shear stress is zero. These planes are termed as principal planes and normal stresses acting on them are known as principal stresses.

Thus , $\sigma_1 = OL$

$\sigma_2 = OM$

3. The maximum shear stress in an element is given by the top and bottom points of the circle i.e by points J_1 and J_2 ,Thus the maximum shear stress would be equal to the radiusof i.e. $\sigma_{max}= 1/2(\sigma_1+ \sigma_2)$,the corresponding normal stress is obviously the distance OP = $1/2 (\sigma_x+\sigma_y)$, Further it can also be seen that the planes on which the shear stress is maximum are situated 90^0 from the principal planes (on circle), and 45^0 in the material.

4. The minimum normal stress is just as important as the maximum. The algebraic minimum stress could have a magnitude greater than that of the maximum principal stress if the state of stress were such that the centre of the circle is to the left of orgin.

i.e. if $\sigma_1 = 20$ MN/m^2 (say)

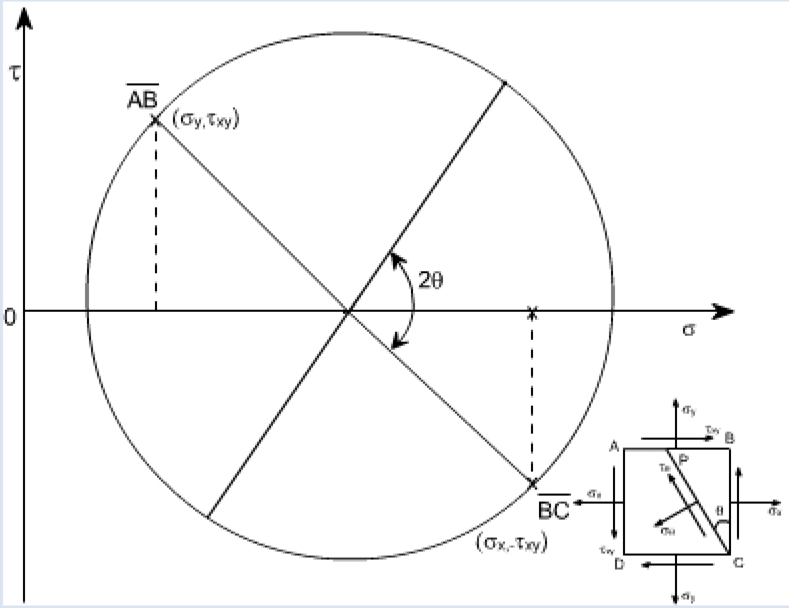
$\sigma_2 = 80$ MN/m^2 (say)

Then $_{max}{}^m = (\sigma_1+\sigma_2 / 2) = 50$ MN/m^2

If should be noted that the principal stresses are considered a maximum or minimum mathematically e.g. a compressive or negative stress is less than a positive stress, irrespective or numerical value.

5. Since the stresses on perpendular faces of any element are given by the co- ordinates oftwo diametrically opposite points on the circle, thus, the sum of the two normal stresses forany and all orientations of the element is constant, i.e. Thus sum is an invariant for any particular state of stress.

Sum of the two normal stress components acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the orientation of these planes.



This can be also understand from the circle Since AB and BC are diametrically opposite thus, whatever may be their orientation, they will always lie on the diametre or we can saythat their sum won't change, it can also be seen from analytical relations

We know $$\sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

on plane BC; = 0

$_{n1} = \sigma_x$

on plane AB; $= 270^0$

$_{n2} = \sigma_y$

Thus $\sigma_{n1} + \sigma_{n2} = \sigma_x + \sigma_y$

1. If $\sigma_1 = \sigma_2$, the Mohr's stress circle degenerates into a point and no shearingstresses are developed on xy plane.

2. If $\sigma_x + \sigma_y = 0$, then the center of Mohr's circle coincides with the origin ofco-ordinates.

CHAPTER-3:-STRESSES IN BEAM AND SHAFTS

Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms

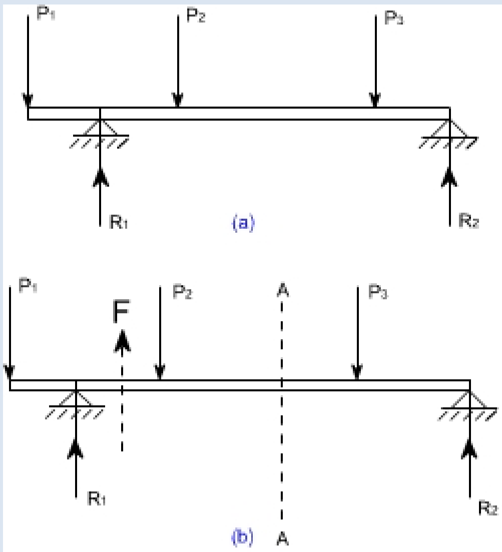


Fig 1

Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2,P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is 'F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces 'F' is as a shear force. The shearing force at any x- section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force 'F' to as follows:

At any x-section of a beam, the shear force 'F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated infigures 2 and 3.

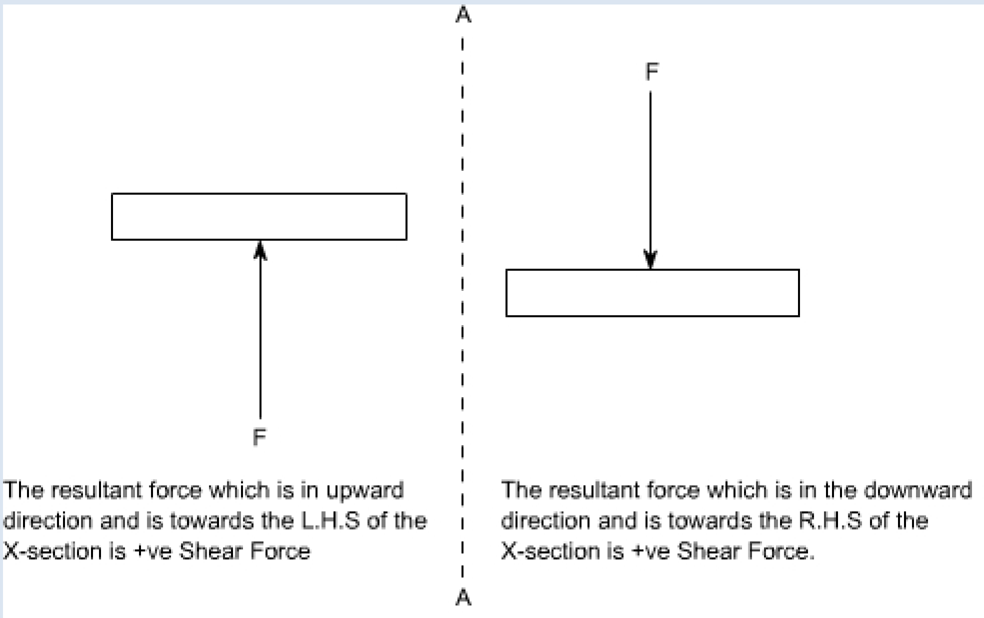


Fig 2: Positive Shear Force

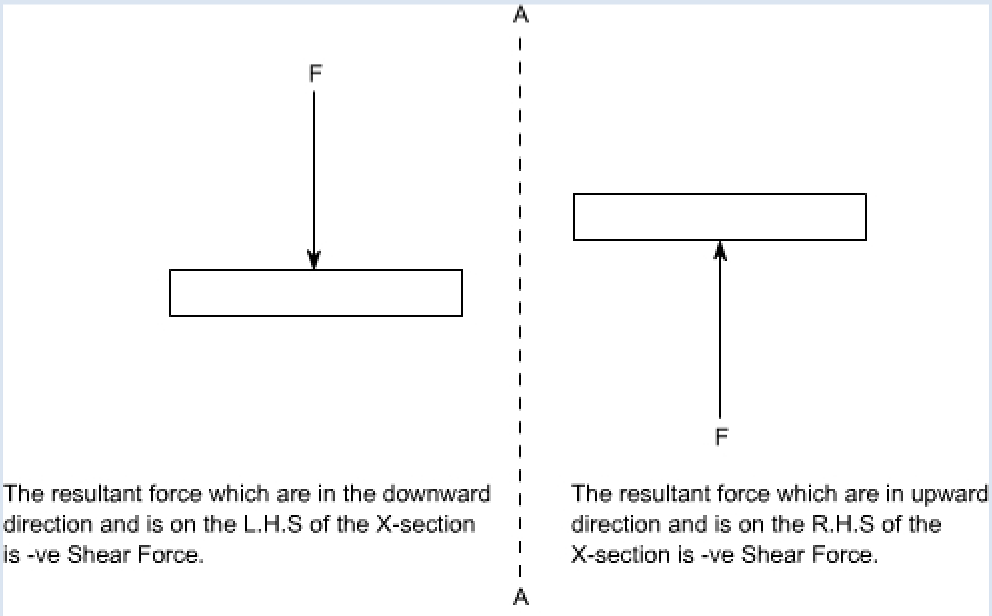


Fig 3: Negative Shear Force

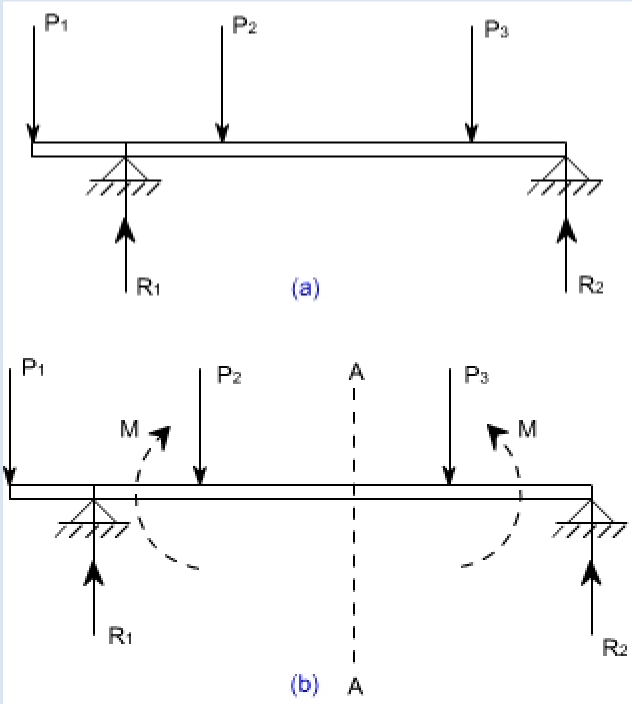


Fig 4

Bending Moment:

Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be 'M' in

C.C.W. Then 'M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as the algebraic sum of the moments about an x-section of all the forces acting on either side of the section

Sign Conventions for the Bending Moment:

For the bending moment, following sign conventions may be adopted as indicated in Fig 5and Fig 6.

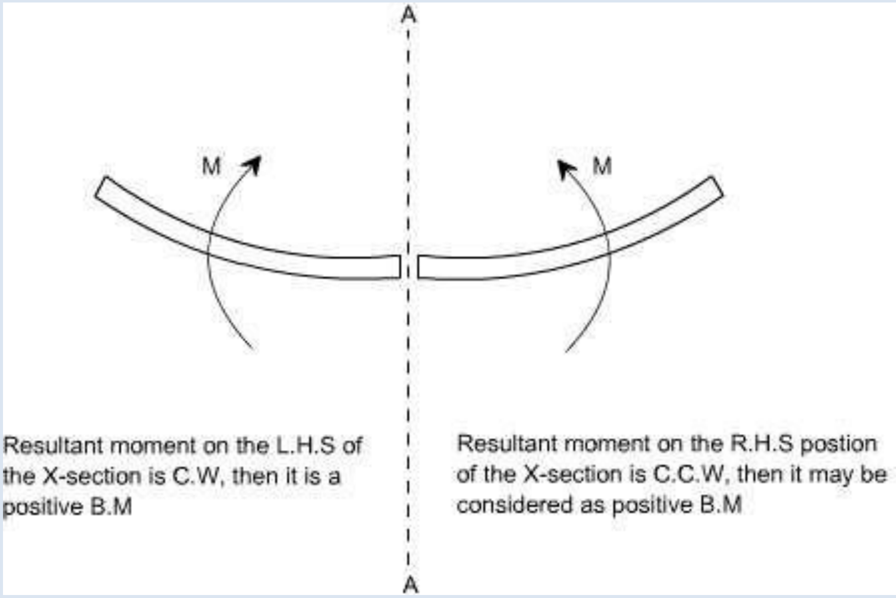


Fig 5: Positive Bending Moment

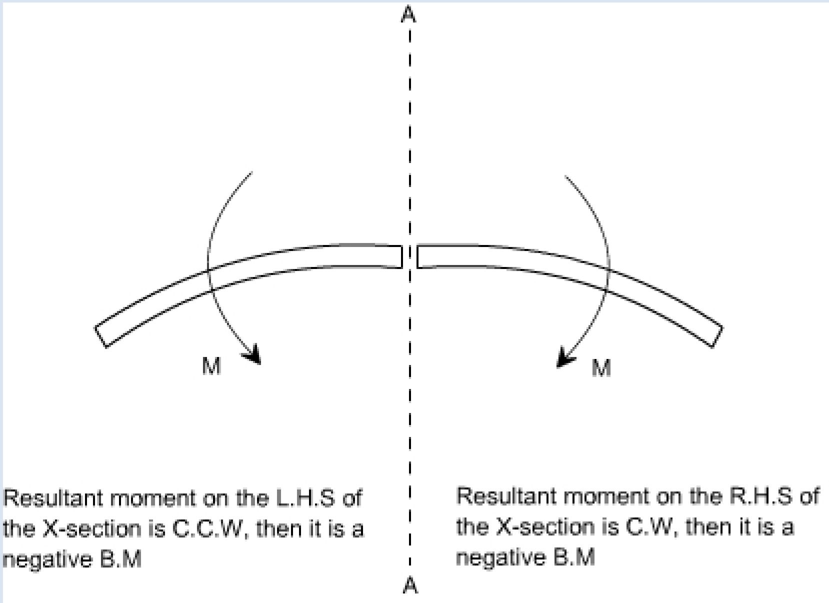


Fig 6: Negative Bending Moment

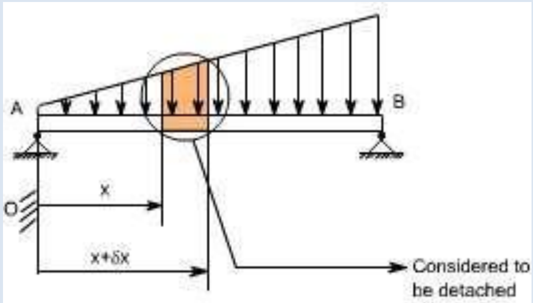
Some times, the terms 'Sagging' and Hogging are generally used for the positive andnegative bending moments respectively.

Bending Moment and Shear Force Diagrams:

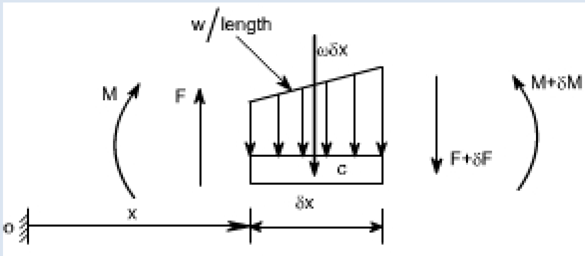
The diagrams which illustrate the variations in B.M and S.F values along the length of thebeam for any fixed loading conditions would be helpful to analyze the beam further.

Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force'F' varies along the length of beam. If x denotes the length of the beam, then F is functionx i.e. F(x). Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment 'M' varies along the length of the beam. Again M is a function x i.e. M(x). **Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:**

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established. Let us consider a simply supported beam AB carrying a uniformly distributed loadw/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam atdistance 'x' from the origin '0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loadedbeam are the following

- The shearing force F and F+ δF at the section x and x + δx respectively.
- The bending moment at the sections x and x + δx be M and M + dM respectively.

• Force due to external loading, if 'w' is the mean rate of loading per unit length then the total loading on this slice of length δx is w. δx, which is approximately acting through the centre 'c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre 'c'. This small element must be in equilibrium under the action of these forces and couples. Now let us take the moments at the point 'c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \to 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \to 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots (2)$$

Conclusions: From the above relations, the following important conclusions may bedrawn

• From Equation (1), the area of the shear force diagram between any two points, from the basic calculus is the bending moment diagram

$$M = \int F.dx$$

• The slope of bending moment diagram is the shear force, thus

$$F = \frac{dM}{dx}$$

Thus, if F=0; the slope of the bending moment diagram is zero and the bending momentis therefore constant.'

- The maximum or minimum Bending moment occurs where $\frac{dM}{dx} = 0.$

The slope of the shear force diagram is equal to the magnitude of the intensity of the distributed loading at any position along the beam. The –ve sign is as a consequence of our particular choice of sign conventions

Procedure for drawing shear force and bending moment diagram:

Preamble:

The advantage of plotting a variation of shear force F and bending moment M in a beam as a function of ‘x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment.

Further, the determination of value of M as a function of ‘x' becomes of paramount importance so as to determine the value of deflection of beam subjected to a given loading.

Construction of shear force and bending moment diagrams:

A shear force diagram can be constructed from the loading diagram of the beam. In order to draw this, first the reactions must be determined always. Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.

When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam. No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations.

The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams giving due regard to sign. The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

It may also be observed that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. It may also further observe that dm/dx= F therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero. In order to check the validity of the bending moment diagram, the terminal conditions for the moment must be satisfied. If the end is free or pinned, the computed sum must be equal to zero. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction. These conditions must always be satisfied.

Illustrative problems:

In the following sections some illustrative problems have been discussed so as to illustrate the procedure for drawing the shear force and bending moment diagrams

1. **A cantilever of length carries a concentrated load 'W' at its free end.**

Draw shear force and bending moment.

Solution:

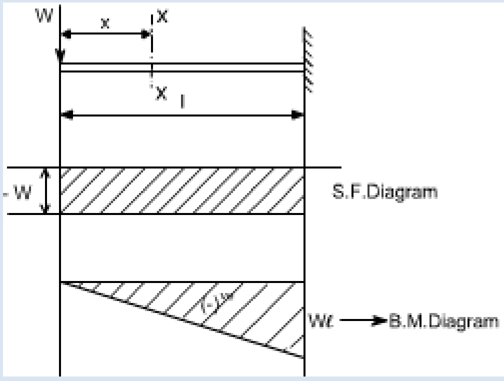
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative

Taking moments about the section gives (obviously to the left of the section)

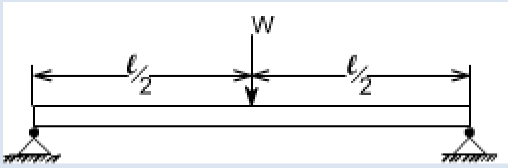
M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention)

so that the maximum bending moment occurs at the fixed end i.e. M = -W l

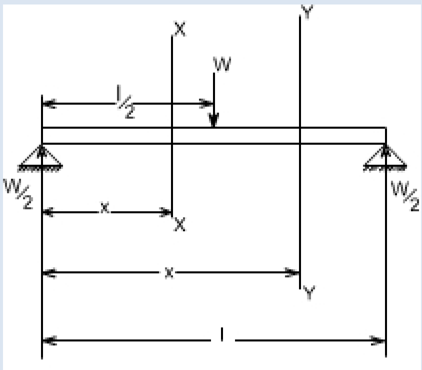
From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



2. Simply supported beam subjected to a central load (i.e. load acting at the mid- way)



By symmetry the reactions at the two supports would be W/2 and W/2. now consider anysection X-X from the left end then, the beam is under the action of following forces.

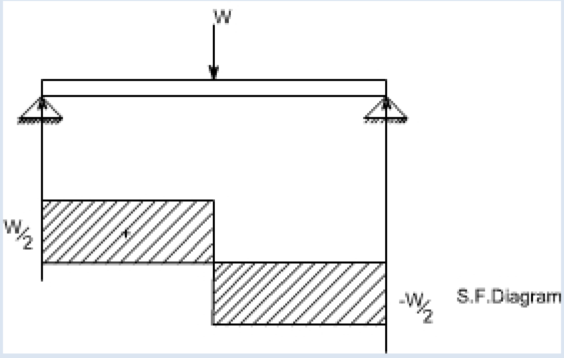


.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

If we consider another section Y-Y which is beyond l/2 then

$$S.F_{Y-Y} = \frac{W}{2} - W = \frac{-W}{2}$$

for all values greater = l/2 Hence S.F diagram can beplotted as,



.For B.M diagram:

If we just take the moments to the left of the cross-section,

$$B.M_{x\text{-}x} = \frac{W}{2}\,x \text{ for } x \text{ lies between } 0 \text{ and } l/2$$

$$B.M_{at\,x=\frac{l}{2}} = \frac{W}{2}\,\frac{l}{2} \text{ i.e } B.M \text{ at } x = 0$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}\,x - W\left(x - \frac{l}{2}\right)$$

Again

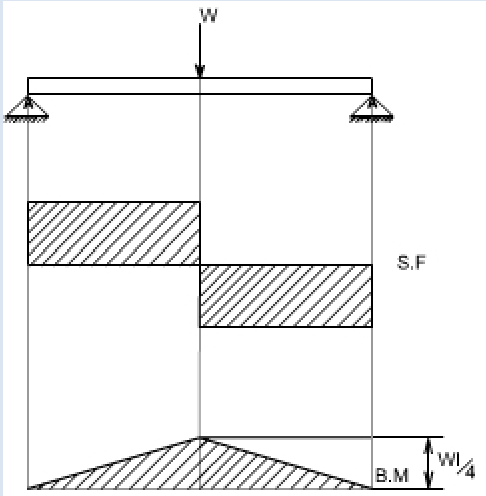
$$= \frac{W}{2}\,x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}\,x + \frac{Wl}{2}$$

$$B.M_{at\,x=l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

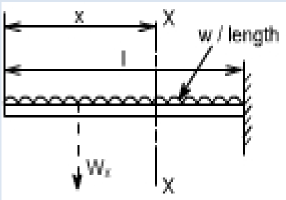
$$= 0$$

Which when plotted will give a straight relation i.e.



It may be observed that at the point of application of load there is an abrupt change inthe shear force, at this point the B.M is maximum.

3. A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity isgiven w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we justtake the resultant of all the forces on the left of the X-section, then

$S.F_{xx}$ = -Wx for all values of ‘x' ------------- **(1)**

$S.F_{xx}$ = 0

$S.F_{xx \text{ at } x=1}$ = -Wl

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

Therefore, the bending moment at any cross-section X-X is

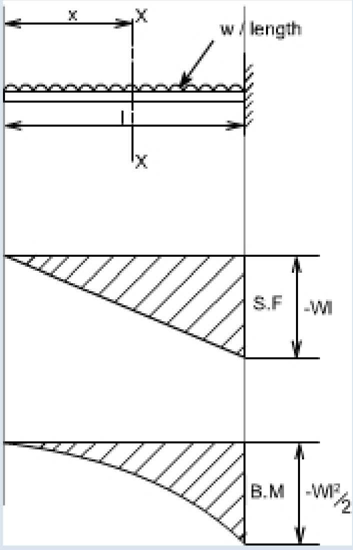
$$B.M_{X\text{-}X} = -Wx\,\frac{x}{2}$$
$$= -W\frac{x^2}{2}$$

The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

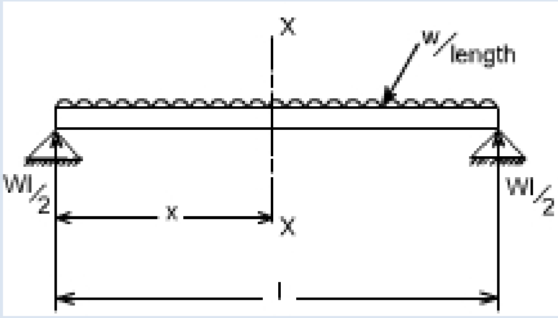
The extreme values of this would be at x = 0 and x = l

$$B.M_{\text{at } x = l} = -\frac{Wl^2}{2}$$
$$= \frac{Wl}{2} - Wx$$

Hence S.F and B.M diagram can be plotted as follows:



4. Simply supported beam subjected to a uniformly distributed load [U.D.L].



The total load carried by the span would be

= intensity of loading x length

= w x l

By symmetry the reactions at the end supports are each wl/2

If x is the distance of the section considered from the left hand end of the beam.

S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$

$$= W\left(\frac{l}{2} - x\right)$$

Giving a straight relation, having a slope equal to the rate of loading or intensity of theloading.

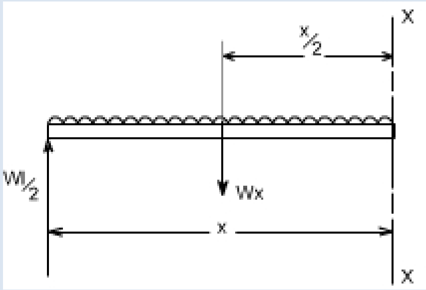
$$S.F_{at\ x=0} = \frac{wl}{2} - wx$$

so at

$$S.F_{at\ x=\frac{l}{2}} = 0 \text{ hence the S.F is zero at the centre}$$

$$S.F_{at\ x=l} = -\frac{Wl}{2}$$

The bending moment at the section x is found by treating the distributed load as acting atits centre of gravity, which at a distance of x/2 from the section



$$B.M_{x\text{-}x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

so the

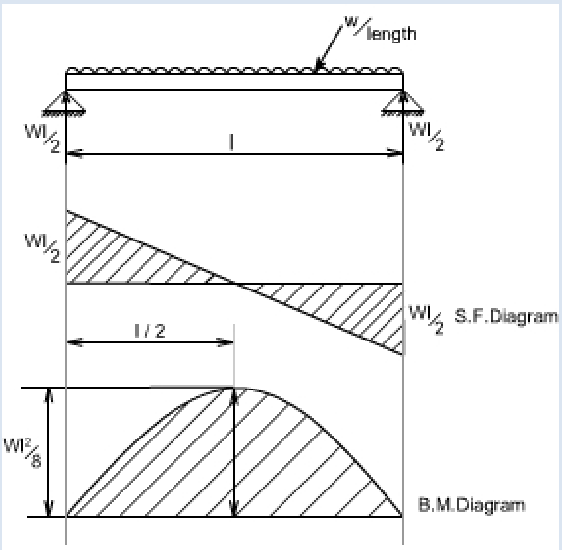
$$= W.\frac{x}{2}(l - 2) \quad \ldots\ldots(2)$$

$$B.M_{at\ x=0} = 0$$

$$B.M_{at\ x=l} = 0$$

$$B.M\Big|_{at\ x=l} = -\frac{Wl^2}{8}$$

So the equation (2) when plotted against x gives rise to a parabolic curve and the shearforce and bending moment can be drawn in the following way will appear as follows:



Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

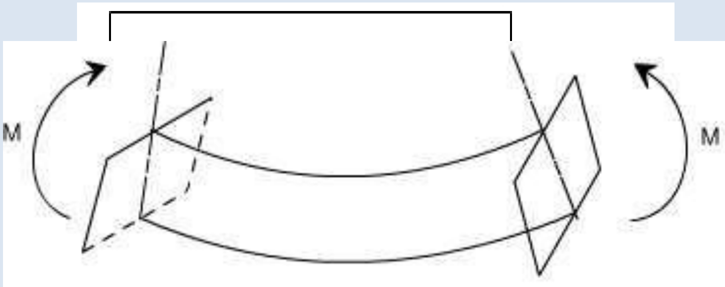
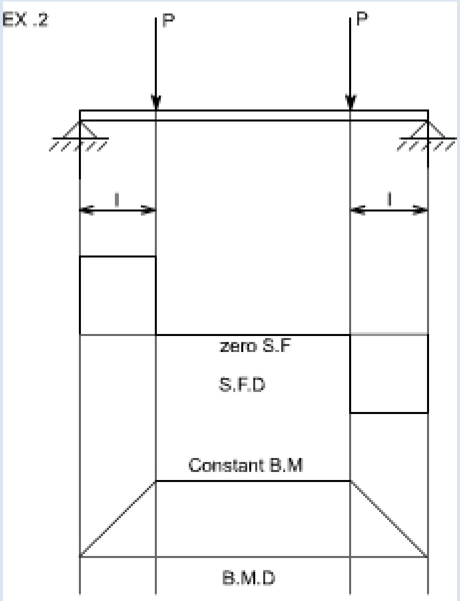
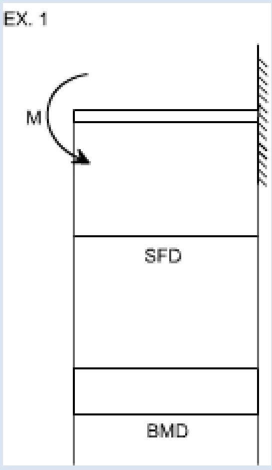


Fig (2)

Fig (1)

When a member is loaded in such a fashion it is said to be in pure bending**.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below :

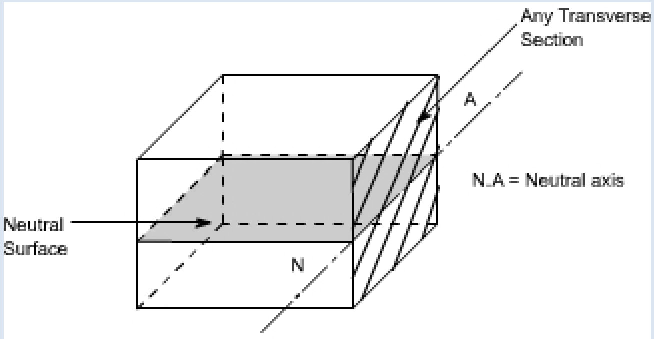




When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,

1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross- section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.

2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.

The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axis Neutral axis **(N A)** .

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e.**H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle .

Consider now fiber AB in the material, at a distance y from the N.A, when the beambends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

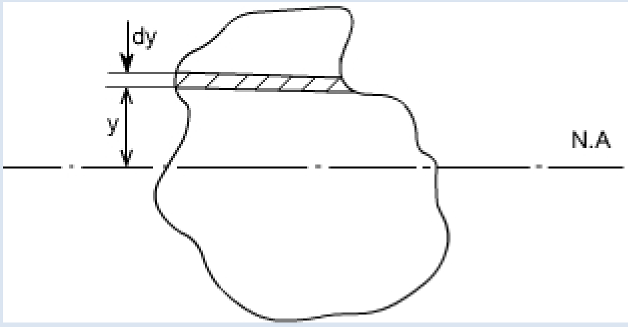
Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \text{.............(1)}$$



Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibreat a distance 'y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area 'dA' then the force on the strip is

$$F = \sigma\ \delta A = \frac{E}{R} y\ \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \Sigma \frac{E}{R} y^2\ \delta A = \frac{E}{R} \Sigma y^2 \delta A$$

Now the term $\Sigma y^2 \delta A$

is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \qquad \text{.........(2)}$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

.

This equation is known as the Bending Theory Equation. The above proof has involved the assumption of pure bending without any shear force being present Therefore this termed as the pure bending equation. This equation gives distribution ofstresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max^m} = \frac{M}{I} y_{max^m}$$

For any given allowable stress the maximum moment which can be accepted by aparticular shape of cross-section is therefore

$$M = \frac{I}{y_{max^m}} \sigma_{max^m}$$

For ready comparison of the strength of various beam cross-section this relationship issome times written in the form

$$M = Z \sigma_{max^m} \text{ where } Z = \frac{I}{y_{max^m}}$$

Is termed as section modulus

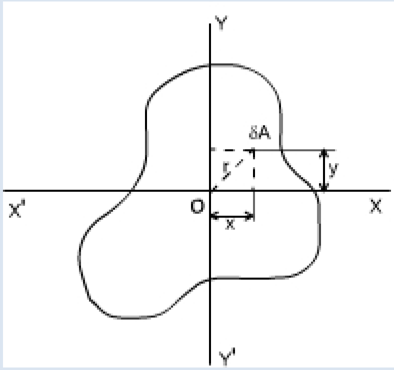
The higher value of Z for a particular cross-section, the higher the bending moment which it can withstand for a given maximum stress.

Theorems to determine second moment of area: There are two theorems which are helpful to determine the value of second moment of area, which is required to be used while solving the simple bending theory equation.

Second Moment of Area :

Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis. (This property arised while we were driving bending theory equation). This is also known as the moment of inertia. An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a

given axis and the second moment being the square of the distance or $\int y^2 . dA$



Consider any cross-section having small element of area d A then by the definition

I_x(Mass Moment of Inertia about x-axis) = $\int y^2\, dA$ and I_y(Mass Moment of Inertia about y-axis) = $\int x^2\, dA$

Now the moment of inertia about an axis through 'O' and perpendicular to the plane of figure is called the polar moment of inertia. (The polar moment of inertia is also the area moment of inertia).

i.e,

J = polar moment of inertia

$$
\begin{aligned}
&= \int r^2 dA \\
&= \int (x^2 + y^2)\, dA \\
&= \int x^2 dA + \int y^2 dA \\
&= I_X + I_Y
\end{aligned}
$$

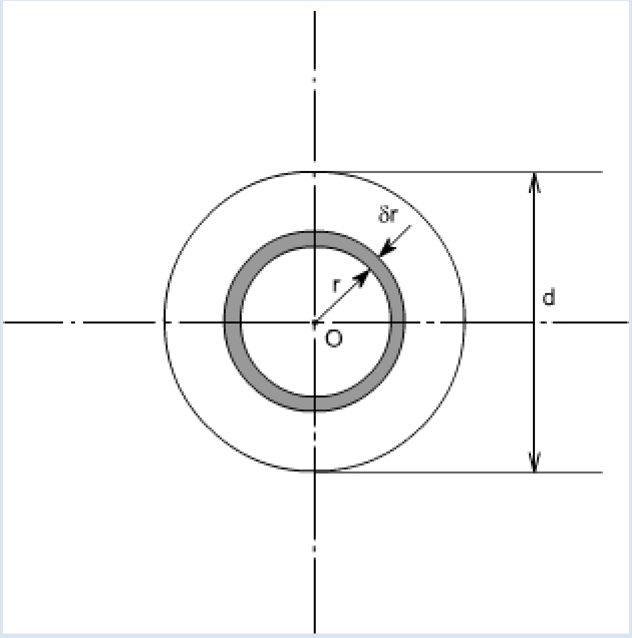
$$\text{or } J = I_X + I_Y \qquad \text{..........} (1)$$

The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION :

For a circular x-section, the polar moment of inertia may be computed in the following manner



Consider any circular strip of thickness r located at a radius 'r'. Thanthe area of the circular strip would be dA = 2 r.   r

$$J = \int r^2 \, dA$$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 \, 2\pi \, \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \, \delta r$$

$$J = 2\pi \left[\frac{r^4}{4} \right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however,by perpendicular axistheorem

$$J = I_x + I_y$$

But for the circular cross-section ,the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

for a hollow circular sectionof diameter D and d, the values of J and I are defined as

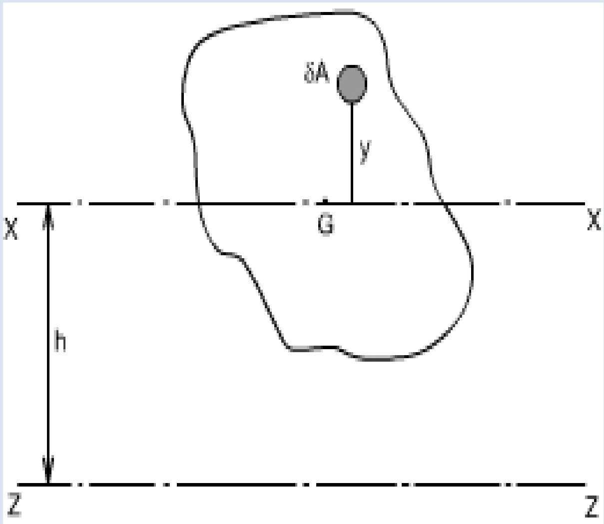
$$J = \frac{\pi\left(D^4 - d^4\right)}{32}$$

$$I = \frac{\pi\left(D^4 - d^4\right)}{64}$$

Thus

Parallel Axis Theorem:

The moment of inertia about any axis is equal to the moment of inertia about a parallelaxis through the centroid plus the area times the square of the distance between theaxes.



If ‘ZZ' is any axis in the plane of cross-section and ‘XX' is a parallel axis through the centroid G, of the cross-section, then

$I_z = \int (y + h)^2 \; dA$ by definition (moment of inertia about an axis ZZ)

$= \int (+ 2yh + h^2) dA$

$= \int y^2 dA + h^2 \int dA + 2h \int y dA$

Since $\int y dA = 0$

$= \int y^2 dA + h^2 \int dA$

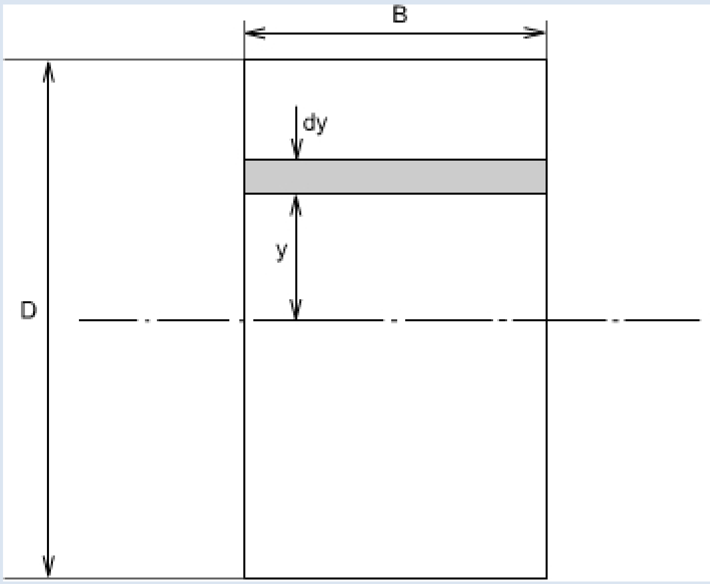
$= \int y^2 dA + h^2 A$

$I_z = I_x + Ah^2 \qquad I_x = I_G$ (since cross-section axes also pass through G)

Where A = Total area of the section

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed asbelow :



Consider the rectangular beam cross-section as shown above and an element of area **dA** , thickness **dy** , breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

$$I_{N.A} = \int y^2 \, dA$$

Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

$$I_{N.A} = \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 (B\,dy)$$

$$= B \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 \, dy$$

$$= B \left[\frac{y^3}{3} \right]_{\frac{D}{2}}^{\frac{-D}{2}}$$

$$= \frac{B}{3} \left[\frac{D^3}{8} - \left(\frac{-D^3}{8} \right) \right]$$

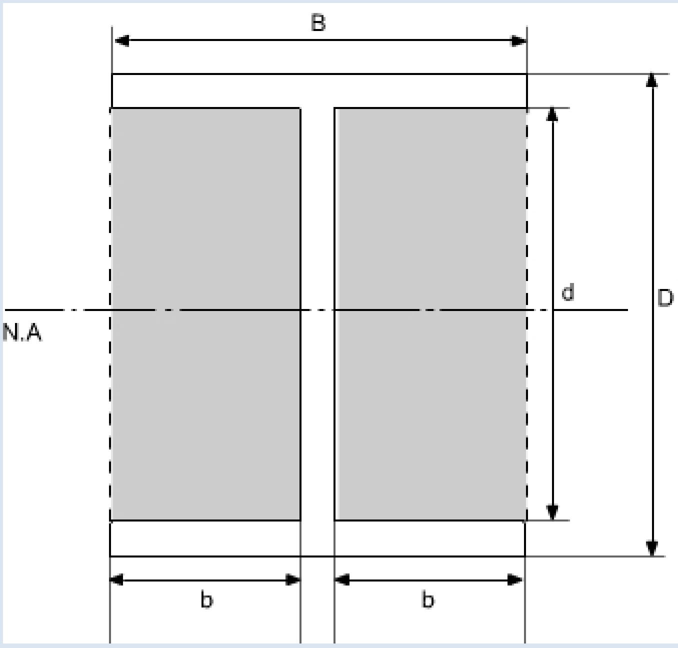
$$= \frac{B}{3} \left[\frac{D^3}{8} + \frac{D^3}{8} \right]$$

$$I_{N.A} = \frac{BD^3}{12}$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of **0** to **D** .

Therefore $$I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3}$$

These standards formulas prove very convenient in the determination of I_{NA} for build up sections which can be conveniently divided into rectangles. For instance if we just want to find out the Moment of Inertia of an I - section, then we can use the above relation.



$$I_{N.A} = I_{\text{of dotted rectangle}} - I_{\text{of shaded portion}}$$

$$\therefore I_{N.A} = \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right)$$

$$I_{N.A} = \frac{BD^3}{12} - \frac{bd^3}{6}$$

Use of Flexure Formula:

Illustrative Problems:

An I - section girder, 200mm wide by 300 mm depth flange and web of thickness is 20mm is used as simply supported beam for a span of 7 m. The girder carries a distributed load of 5 KN /m and a concentrated load of 20 KN at mid-span.

Determine the

(i). The second moment of area of the cross-section of the girder (ii). The maximum stress set up.

Solution:

The second moment of area of the cross-section can be determined as follows :

For sections with symmetry about the neutral axis, use can be made of standard I value for a rectangle about an axis through centroid i.e. (bd 3)/12. The section can thus be divided into convenient rectangles for each of which the neutral axis passes through the centroid. Example in the case enclosing the girder by a rectangle

$$I_{girder} = I_{rectangle} - I_{shaded\ portion}$$

$$= \left[\frac{200 \times 300^3}{12}\right] 10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right] 10^{-12}$$

$$= (4.5 - 2.64)10^{-4}$$

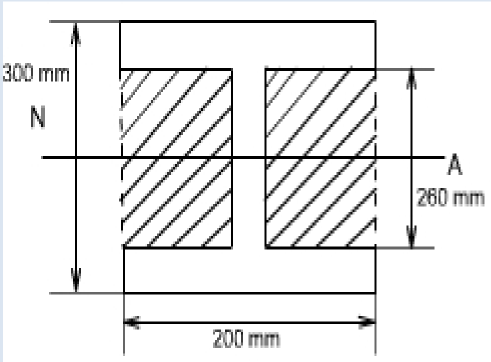
$$= 1.86 \times 10^{-4}\ m^4$$

The maximum stress may be found from the simple bending theory by equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

i.e.

$$\sigma_{max^m} = \frac{M_{max^m}}{I}\, y_{max^m}$$



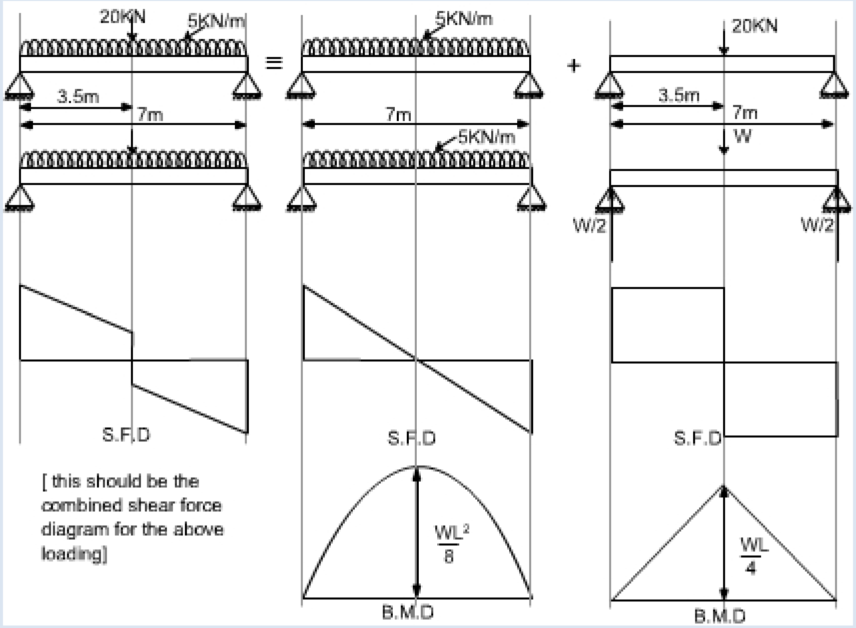
__Computation of Bending Moment:__

In this case the loading of the beam is of two types

(a) Uniformly distributed load

(b) Concentrated Load

In order to obtain the maximum bending moment the technique will be to consider each loading on the beam separately and get the bending moment due to it as if no otherforces acting on the structure and then superimpose the two results.



Hence

$$
\begin{aligned}
M_{max^m} &= \frac{wL}{4} + \frac{wL^2}{8} \\
&= \frac{20\times10^3\times7}{4} + \frac{5\times10^3\times7^2}{8} \\
&= (35.0 + 30.63)10^3 \\
&= 65.63\ kNm \\
\sigma_{max^m} &= \frac{M_{max^m}}{I}\ y_{max^m} \\
&= \frac{65.63\times10^3\times150\times10^3}{1.06\times10^{-4}} \\
\sigma_{max^m} &= 51.8\ MN/m^2
\end{aligned}
$$

Shearing Stresses in Beams

All the theory which has been discussed earlier, while we discussed the bending stresses in beams was for the case of pure bending i.e. constant bending moment acts along the entire length of the beam.

CHAPTER-4:- SLOPE AND DEFLECTION OF BEAMS

Introduction:

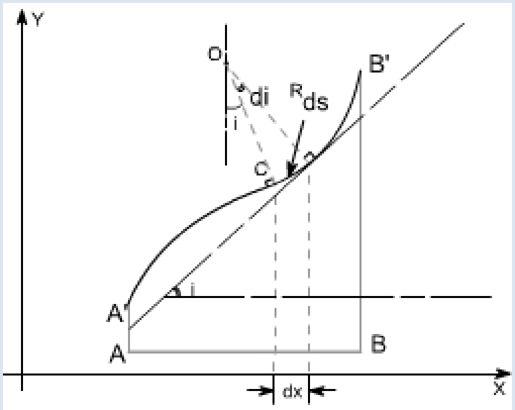
In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.
2. The curvature is always small.
3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small andhence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Further, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y

– axis shows the deflection.

Further,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di

But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$ds = R\,di$

however,

$ds = dx$ [usually for small curvature]

Hence

$ds = dx = R\,di$

or $\boxed{\frac{di}{dx} = \frac{1}{R}}$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationshipamong shear force, bending moment and deflection of the beam may be obtained as Differentiating the equation as derived

$$\frac{dM}{dx}=EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx}=F$$

Thus,

$$F=EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity ofloading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' isthe deflection of the loaded beam,
then the following important relations can be arrived at

$$\boxed{\text{slope} = \frac{dy}{dx}}$$

$$\boxed{\text{B.M} = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods.The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A \text{ - - - - this equation gives the slope of the loaded beam.}$$

Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}dx + Ax + B$$

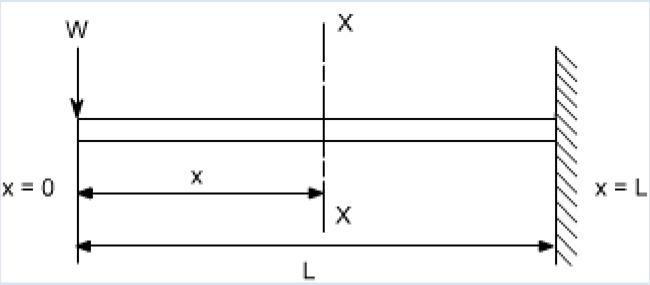
Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples : let us consider few illustrative examples to have a familiarty with the direct integration method

Case 1:

Cantilever Beam with Concentrated Load at the end:-

A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from theleft end or the reference, and write down the expressions for the shear force abd the bending moment

$S.F|_{x-x} = -W$

$B.M|_{x-x} = -W.x$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}\,dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}\,dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundaryconditions as defined below

i.e at x= L ; y= 0 ____________________(1)at

x = L ; dy/dx = 0 ---------------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

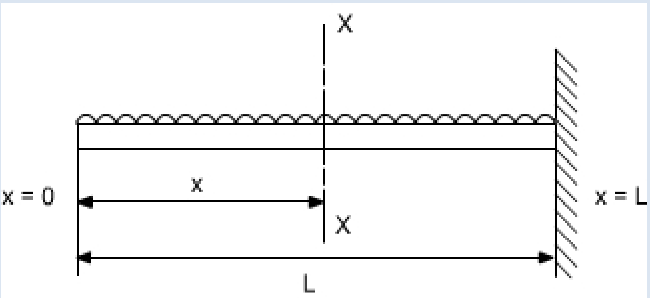
$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max}m = +\frac{WL^2}{2EI}}$$

<u>Case 2:</u>

<u>A Cantilever with Uniformly distributed Loads:-</u>

In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w /length.The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int \frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0
2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

So $y_{max}m$ will be at $x = 0$

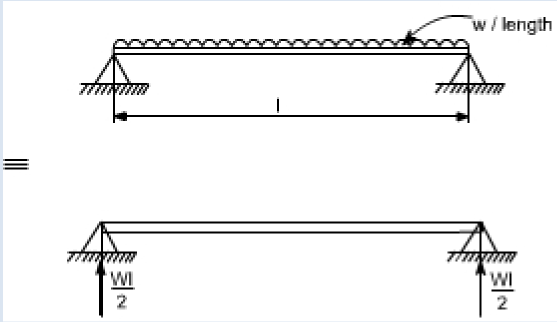
$$\boxed{y_{max}m = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{max}m = \frac{wL^3}{6EI}}$$

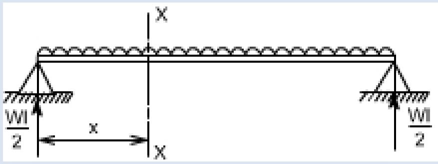
<u>Case 3:</u>

<u>Simply Supported beam with uniformly distributed Loads:-</u>

In this case a simply supported beam is subjected to a uniformly distributed load whoserate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----}(1)$$

Boundary conditions which are relevant in this case are that the deflection at eachsupport must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundaryconditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2Conclusions

Then $$y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = \frac{5wL^4}{384EI}}$$

(i) The value of the slope at the position where the deflection is maximum would be zero.

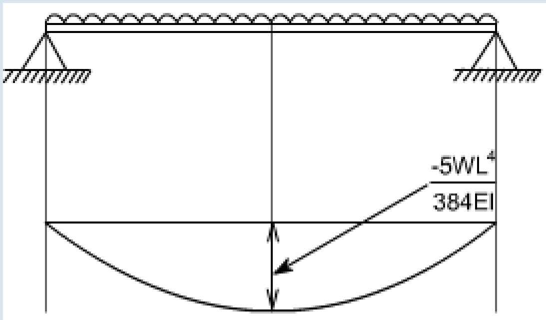
(ii) Thevalue of maximum deflection would be at the centre i.e. at x = L/2. The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shearforce and rate of loading.

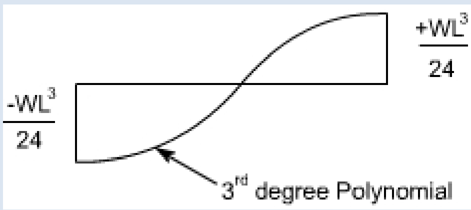
Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



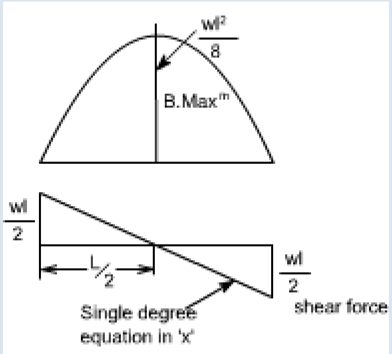
Slope (dy/dx)

$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



Bending Moment So the bending moment diagram would

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$ be



Shear Force

Shear force is obtained by taking third derivative.

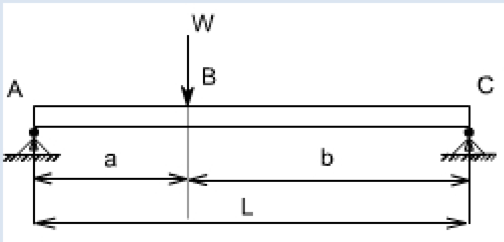
$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

Rate of intensity of loading

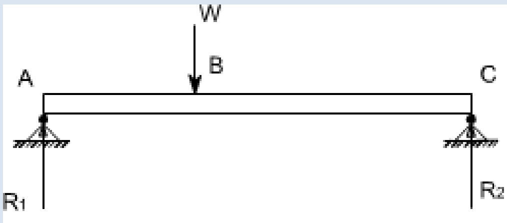
$$EI\frac{d^4y}{dx^4} = -w$$

Case 4:

The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \le x \le a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x - a) \quad a \le x \le l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1\,x$$

$$EI\frac{d^2y}{dx^2} = R_1\,x - W\,(x - a)$$

These two equations can be integrated in the usual way to find ‘y' but this will result infour constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at $x = 0$; $y = 0$ in the portion AB i.e. $0 \le x \le a$

(b) at $x = l$; $y = 0$ in the portion BC i.e. $a \le x \le l$

(c) at $x = a$; dy/dx, the slope is same for both portion

(d) at $x = a$; y, the deflection is same for both portion By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{--------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{------}(3)$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{-------}(4)$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{------}(5)$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{------}(6)$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y|_{\text{from equation 5}} = y|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0;$

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{----}\mathbf{for\ 0 \le x \le a}\text{-----(7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{For } \mathbf{for\ a \le x \le l}\text{----- (8)}$$

so either of the equation (7) or (8) may be used to find the deflection at x = a

hence substituting x = a in either of the equation we get

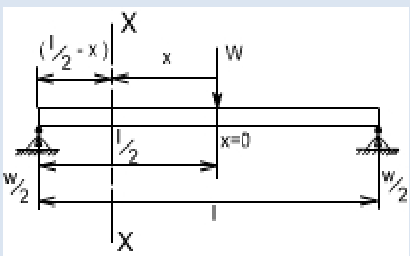
$$Y|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if $a = b = l/2$

$$\boxed{Y_{\max m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD:

There is also an alternative way to attempt this problem in a more simpler way. Let usconsidering the origin at the point of application of the load,



$$S.F|_{xx} = \frac{W}{2}$$
$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$
$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$
$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we havetaken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$
$$B = -\frac{WL^3}{48}$$

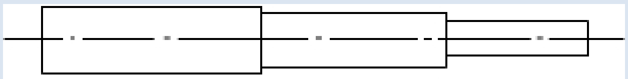
Hence he equation which governs the deflection would be

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$$\boxed{\begin{array}{lll} Y_{max^m}\Big|_{at\,x=0} & = -\dfrac{WL^3}{48EI} & \text{At the centre} \\ \left(\dfrac{dy}{dx}\right)_{max^m}\Big|_{at\,x=\pm\frac{L}{2}} & = \pm\dfrac{WL^2}{16EI} & \text{At the ends} \end{array}}$$

Hence the integration method may be bit cumbersome in some of the case. Anotherlimitation of the method would be that if the beam is of non uniform cross section,



i.e. it is having different cross-section then this method also fails. Sothere are other methods by which we find the deflection like

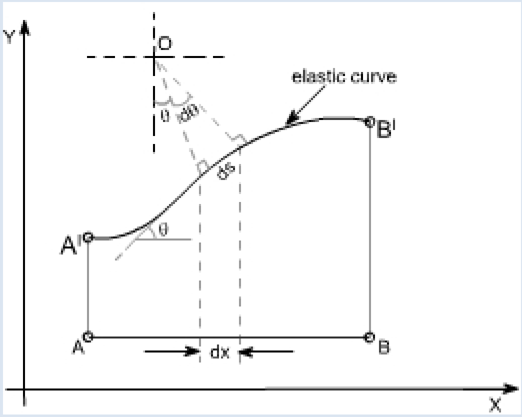
1. Macaulay's method in which we can write the different equation for bending moment for different sections.

2. Area moment methods

MOMENT-AREA METHODS:

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that d is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.

We can assume,

ds = dx [since the curvature is small] hence,R d

= ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

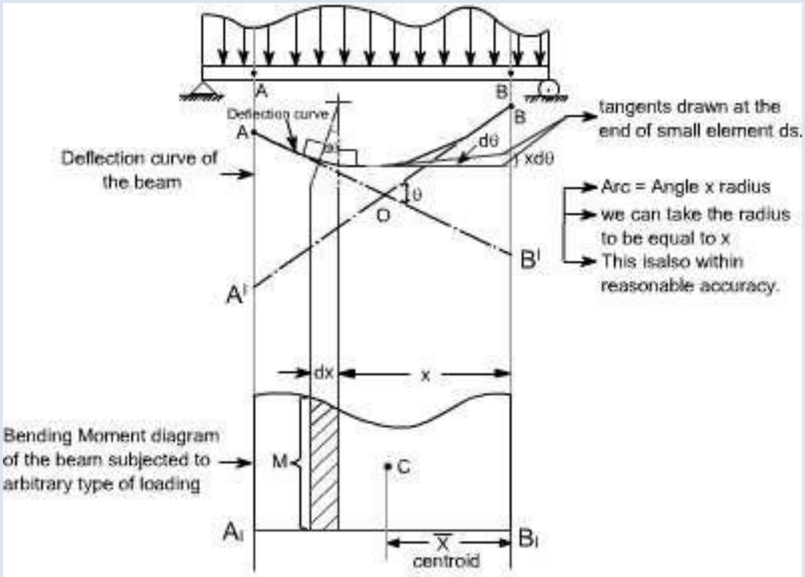
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle,slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$,hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting ds ≈ dx]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \text{ - - - - - (1)}$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of theloaded beam and $A_1B_1$is its corresponding bending moment diagram.

Let AO = Tangent drawn at A BO

= Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are futher understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be d . Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of theshaded bending moment diagram divided by EI.

From the above relationship the total angle between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\begin{Bmatrix} \text{slope or } \theta \\ \text{between any two points} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{EI} \times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram} \end{Bmatrix}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but thevertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x d [each of this intercept may be considered as the arc of a circle of radius x subtended by the angle]

Hence the total distance B'B becomes $\delta = \int_A^B x\, d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of d = M dx / EI as derived earlier

$$\delta = \int_A^B x \frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$$

[This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Therefore,we are in a position to state the above conclusion in the form of theorem as follows:

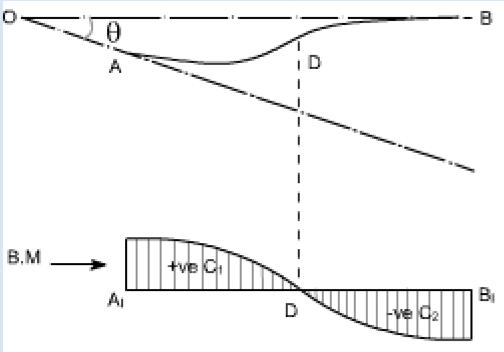
Theorem II:

Deflection of point 'B' relative to point A $= \dfrac{1}{EI} \times \begin{Bmatrix} \text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram} \end{Bmatrix}$ Futher, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $\boxed{\delta_A = \dfrac{1}{EI} A\bar{X}}$

Therefore,the first moment of area may be obtained simply as a product of the total area of the B.M diagram betweenthe points A and B multiplied by the distance to its centroid C. $\bar{x}$

If there exists an inflection point or point of contrefiexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids $C_1$and C_2.Then to find an angle between the tangentsat the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

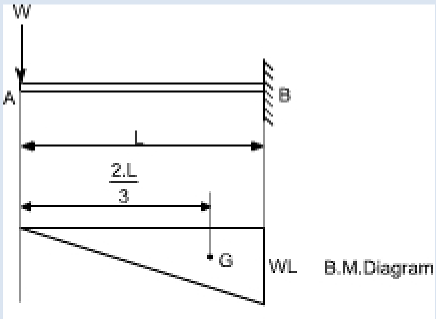
$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Illustrative Examples: Let us study few illustrative examples, pertaining to the use ofthese theorems

Example 1:

1. A cantilever is subjected to a concentrated load at the free end.It is required to find outthe deflection at the free end.

Fpr a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area -moment theorem

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B.M diagram between the points A and B}]$$
$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$
$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area -moment theorem

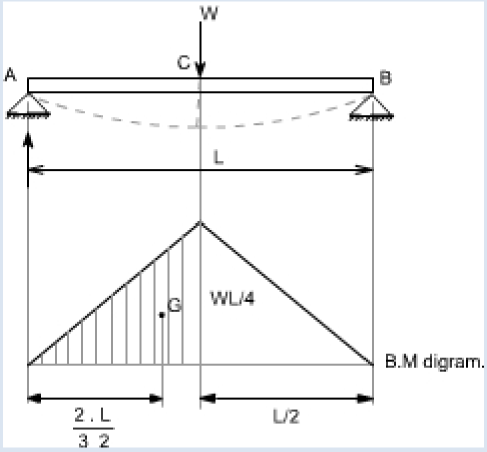
NOTE: In this case the point B is at zero slope.

Thus,

$$\delta = \frac{1}{EI}[\text{first moment of area of B.M diagram between A and B about A}]$$
$$= \frac{1}{EI}[A\bar{y}]$$
$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$
$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid spandetermine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. Thebending moment diagram is drawn below the loaded beam.



Again working relative to the zero slope at the centre C.

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B. M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

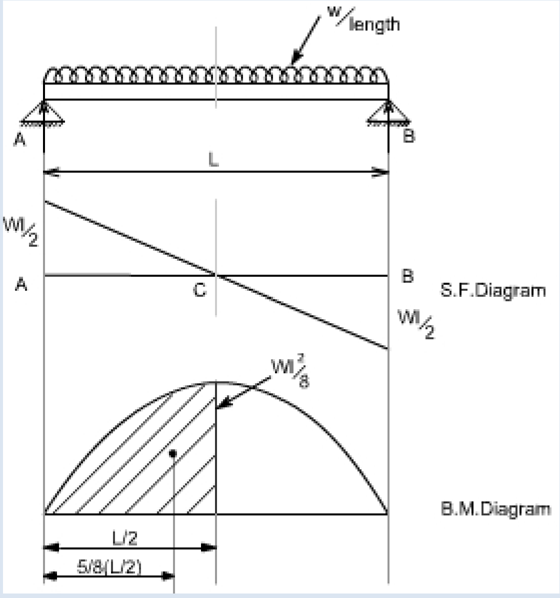
$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, witha intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam, the value ofmaximum B.M is equal to Wl^2 / 8



So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}[\text{Area of B.M diagram between point A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}[A\,\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

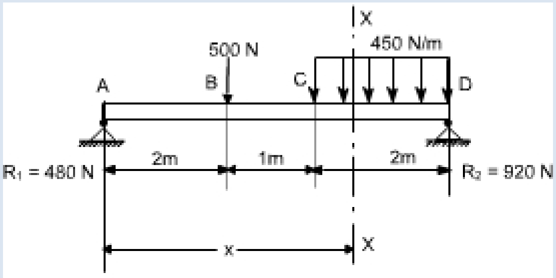
Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially thesame.

For example consider the beam shown in fig below:

Let us write the general moment equation using the definition $M = (\sum M)_L$, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows

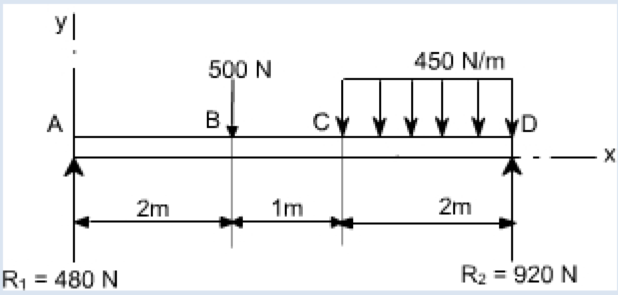


$$M_{AB} = 480\,x\ \text{N.m}$$

$$M_{BC} = \left[480\,x - 500\,(x-2)\right]\text{N.m}$$

$$M_{CD} = \left[480\,x - 500\,(x-2) - \frac{450}{2}(x-3)^2\right]\text{N.m}$$

It may be observed that the equation for M_{CD} will also be valid for both M_{AB} and M_{BC} provided that the terms (x - 2) and (x - 3)^{2}are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and (x - 3)^2 are nonexistent for values of x for which the terms in parentheses are negative.

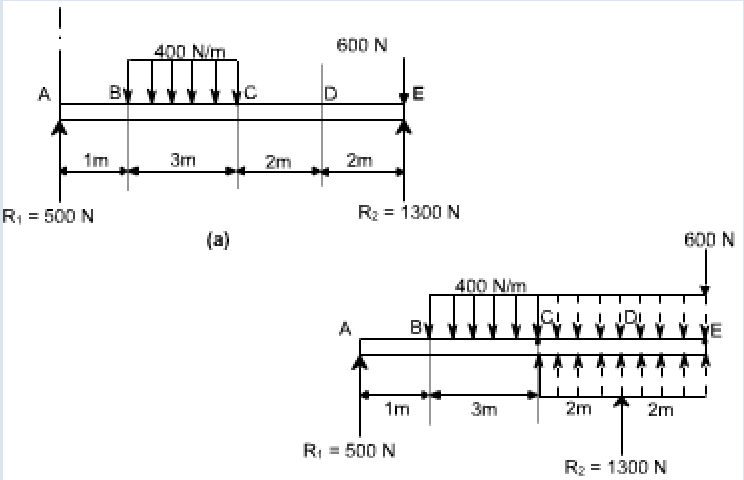


As an clear indication of these restrictions,one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

$$M = \left(480x - 500\,(x-2) - \frac{450}{2}(x-3)^2\right)\text{N.m}$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As an another example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward- distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:



$$M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) N.m$$

It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratary just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

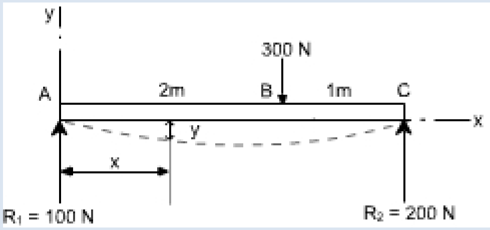
<u>Procedure to solve the problems</u>

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

llustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig.Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution : writing the general moment equation for the last portion BC of the loadedbeam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \quad ....(3)$$

To evaluate the two constants of integration. Let us apply the following boundaryconditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these valuesin Eq. (3) we find C_2 = 0.keep in mind that< x -2 >^{3} is to be neglected for negative values.

2. At the other support where x = 3m, the value of deflection y is also zero. substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50(3-2)^3 + 3.C_1\right) \text{or } C_1 = -133N.m^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) torewrite the slope and deflection equations in the conventional form for the two portions.

segment AB ($0 \le x \le 2m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right)N.m^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right)N.m^3 \qquad ....(5)$$

segment BC ($2m \le x \le 3m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 150\left(x-2\right)^2 - 133x\right)N.m^2 \; ....(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\left(x-2\right)^3 - 133x\right)N.m^3 \; ......(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

We obtain

$50\ x^2 – 133 = 0$ or $x = 1.63$ m (It may be kept in mind that if the solution of the equationdoes not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max^m} = -145\,N.m^3 \qquad ......(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by , the use of y may be reserved to indicate a directed value of deflection.

if E = 30 Gpa and I = 1.9 x 10^6 mm^4 = 1.9 x 10^{-6} m^4 , Eq. (h) becomes

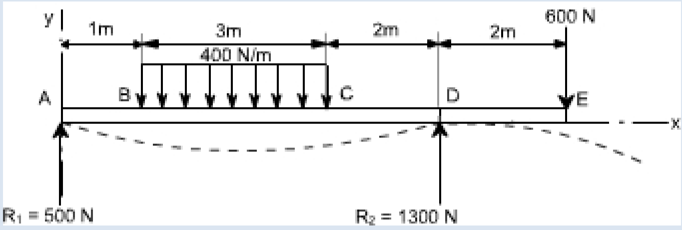
$$y\big|_{max^m} = \left(30\times10^9\right)\left(1.9\times10^{-6}\right)$$

Then

$$= -2.54\,mm$$

Example 2:

It is required to determine the value of EIy at the position midway between thesupports and at the overhanging end for the beam shown in figure below.



Solution:

Writing down the moment equation which is valid for the entire span of the beam and applying the differential equation of the elastic curve, and integrating it twice, we obtain

$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) N.m$$

$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}(x-1)^3 + \frac{200}{3}(x-4)^3 + 650(x-6)^2 + C_1\right) N.m$$

$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}(x-1)^4 + \frac{50}{3}(x-4)^4 + \frac{650}{3}(x-6)^3 + C_1x + C_2\right) N.m^3$$

To determine the value of C_2, It may be noted that EIy = 0 at x = 0,which gives C_2 = 0.Note that the negative terms in the pointed brackets are to be ignored Next,let us usethe condition that EIy = 0 at the right support where x = 6m.This gives

$$0 = \frac{250}{3}(6)^3 - \frac{50}{3}(5)^4 + \frac{50}{3}(2)^4 + 6C_1 \quad \text{or} \quad C_1 = -1308 N.m^2$$

Finally, to obtain the midspan deflection, let us substitute the value of x = 3m in the deflection equation for the segment BC obtained by ignoring negative values of the bracketed terms $x - 4\ ^4$ and $x - 6\ ^3$. We obtain

$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941 \ N.m^3$$

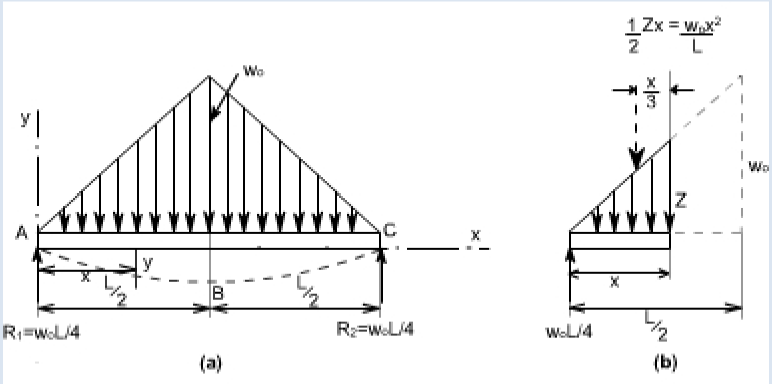
For the overhanging end where x=8 m,we have

$$EIy = \left(\frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)\right)$$

$$= -1814 N.m^3$$

Example 3:

A simply supported beam carries the triangularly distributed load as shown in figure. Determine the deflection equation and the value of the maximum deflection.



Solution:

Due to symmetry, the reactionsis one half the total load of $1/2w_0L$, or $R_1 = R_2 = 1/4w_0L$.Due to the advantage of symmetry to the deflection curve from A to B is the mirror image of that from C to B. The condition of zero deflection at A and of zero slope at B do not require the use of a general moment equation. Only the moment equation for segment AB is needed, and this may be easily written with the aid of figure(b).

Taking into account the differential equation of the elastic curve for the segment AB and integrating twice, one can obtain

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{w_0L}{4}x - \frac{w_0x^2}{L}.\frac{x}{3} \qquad ......(1)$$

$$EI\frac{dy}{dx} = \frac{w_0Lx^2}{8} - \frac{w_0x^4}{12L} + C_1 \qquad ......(2)$$

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} + C_1x + C_2 ......(3)$$

In order to evaluate the constants of integration,let us apply the B.C'swe note that at the support A, $y = 0$ at $x = 0$.Hence from equation (3), we get $C_2 = 0$. Also,because of symmetry, the slope $dy/dx = 0$ at midspan where $x = L/2$.Substituting these conditions in equation (2) we get

$$0 = \frac{w_0L}{8}\left(\frac{L}{2}\right)^2 - \frac{w_0}{12L}\left(\frac{L}{2}\right)^4 + C_1C_1 = -\frac{5w_0L^3}{192}$$

Hence the deflection equation from A to B (and also from C to B because of symmetry) becomes

$$EIy = \frac{w_0 L x^3}{24} - \frac{w_0 x^5}{60L} - \frac{5 w_0 L^3 x}{192}$$

Whichreducesto

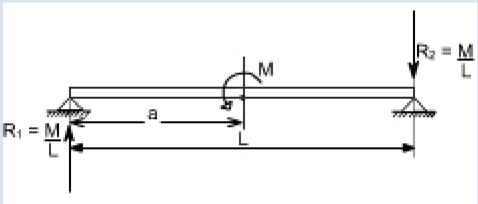
$$EIy = -\frac{w_0 x}{960L}\left(25L^4 - 40L^2x^2 + 16x^4\right)$$

The maximum deflection at midspan,where $x = L/2$ is then found to be

$$\boxed{EIy = -\frac{w_0 L^4}{120}}$$

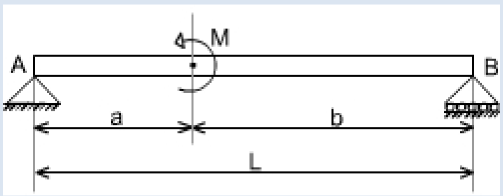
Example 4: couple acting

Consider a simply supported beam which is subjected to a couple M at adistance 'a' fromthe left end. It is required to determine using the Macauley's method.



To deal with couples, only thing to remember is that within the pointed brackets we haveto take some quantity and this should be raised to the power zero.i.e. M x - a ^{0} . We have taken the power 0 (zero) ' because ultimately the term M x - a ^{0}Should have the moment units.Thus with integration the quantity x - a becomes either x - a ^{1}or x - a ^{2}

Or



Therefore, writing the general moment equation we get

$$M = R_1 x - M\langle x - a\rangle \text{ or } EI\frac{d^2y}{dx^2} = M$$

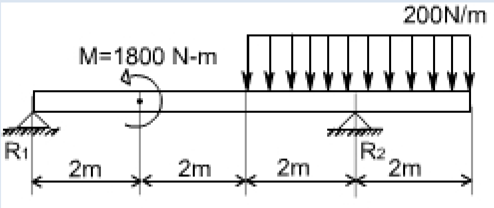
Integrating twice we get

$$EI\frac{dy}{dx} = R_1 . \frac{x^2}{2} - M\langle x - a\rangle^1 + C_1$$

$$EI.y = R_1 . \frac{x^3}{6} - \frac{M}{2}\langle x - a\rangle^2 + C_1 x + C_2$$

Example 5:

A simply supported beam is subjected to U.d.l in combination with couple M. It isrequired to determine the deflection.



This problem may be attemped in the some way. The general moment equation my bewritten as

$$M(x) = R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle\langle x-4\rangle}{2} + R_2\langle x-6\rangle$$

$$= R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle^2}{2} + R_2\langle x-6\rangle$$

Thus,

$$EI\frac{d^2y}{dx^2} = R_1 x - 1800\langle x-2\rangle^0 - \frac{200\langle x-4\rangle^2}{2} + R_2\langle x-6\rangle$$

Integrate twice to get the deflection of the loaded beam.

CHAPTER-5:-COLUMNS AND STRUTS

Introduction:

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail bycrushing when the yield stress of the material in compression is exceeded.

Struts:

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons.

(a). the strut may not be perfectly straight initially.

(b). the load may not be applied exactly along the axis of the Strut.

(c). one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.]

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Bucklingoccurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and isThe quantity I may be written as $I = Ak^2$,

Where I = area of moment of inertia A =

area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k. Therewill be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

Is called the slenderness ratio. It's numerical value indicates whether the member fallsinto the class of columns or struts.

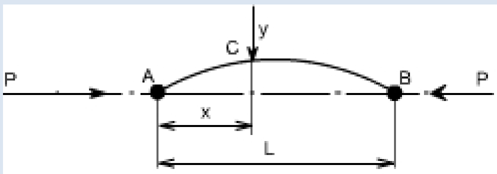
Euler's Theory : The struts which fail by buckling can be analyzed by Euler's theory. Inthe following sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

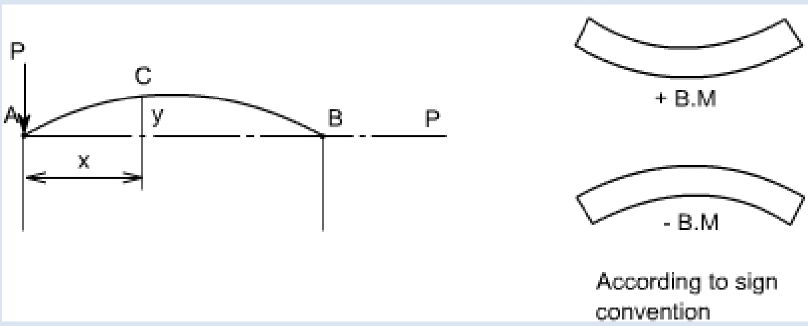
Consider an axially loaded strut, shown below, and is subjected to an axial load ‘P' thisload ‘P' produces a deflection ‘y' at a distance ‘x' from one end.

Assume that the ends are either pin jointed or rounded so that there is no moment ateither end.

Assumption:



The strut is assumed to be initially straight, the end load being applied axiallythrough centroid.



$$B.M|_C = -Py$$

Futher,we know that

$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = -P.y = M$$

In this equation 'M' is not a function 'x'. Therefore this equation can not be integrateddirectly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI\frac{d^2y}{dx^2} + Py = 0$$

Though this equation is in 'y' but we can't say at this stage where the deflectionwould be maximum or minimum.

So the above differential equation can be arranged in the following form $\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$

Let us define a operator D = d/dx

$(D^2 + n^2)$ y =0 where n^2 = P/EI

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus y = A cos (nx) + B sin (nx) Where A and B are some constants.

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$$

Therefore

In order to evaluate the constants A and B let us apply the boundary conditions,(i)at x = 0; y = 0

(ii) at x = L ; y = 0

Applying the first boundary condition yields A = 0.

Applying the second boundary condition gives

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Thus either $B = 0$, or $\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$

if B=0, that y0 for all values of x hence the strut has not buckled yet. Therefore, the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \quad \text{or} \quad \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \quad \text{or} \quad P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the " **Euler Crippling Load** " P_e from which w obtain.

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0 \qquad \text{i.e. } \sin nL = 0$$

The interpretation of the above analysis is that for all the values of the load P, other than those which make sin nL = 0; the strut will remain perfectly straight since

$y = B \sin nL = 0$

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

$\sin nL = 0 \quad \text{or } nL = \pi$

Therefore $n = \frac{\pi}{L}$

Hence $y = B \sin nx = B \sin \frac{\pi x}{L}$

Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that 'L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

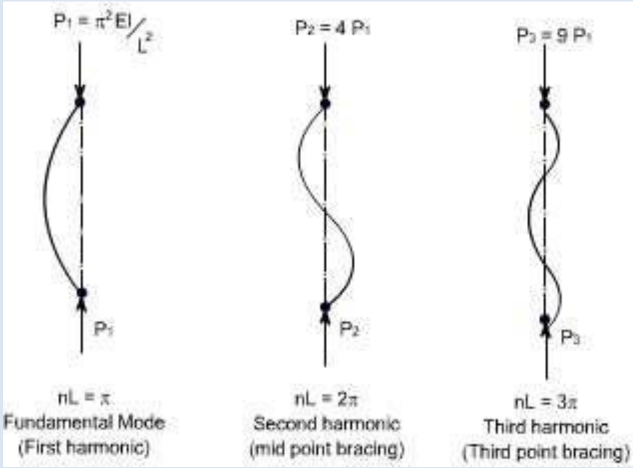
Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; likewise it will be found that the maximum stress is not proportional to load.

The solution chosen of nL is just one particular solution; the solutions nL= 2 , 3 , 5 etc are equally valid mathematically and they do, infact, produce values of

'P_e' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling.

The value selected above is so called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.

The solution nL = 2 produces buckling in two half – waves, 3 in three half-waves etc.



$$L\sqrt{\frac{P}{EI}} = \pi \quad \text{or} \quad P_1 = \frac{\pi^2 EI}{L^2}$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \text{ or } P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

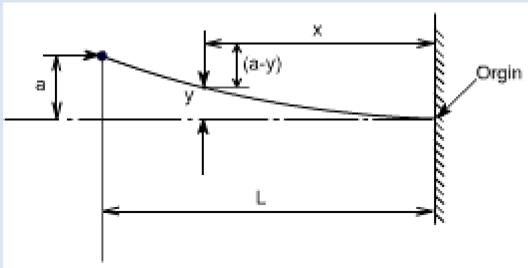
$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \text{ or } P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse.

<u>struts and columns with other end conditions:</u>

Let us consider the struts and columns having different end conditions

Case b: One end fixed and the other free:



writing down the value of bending moment at the point C

$$B.M|_c = P(a-y)$$

Hence, the differential equation becomes,

$$EI\frac{d^2y}{dx^2} = P(a-y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

Let $\frac{P}{EI} = n^2$

Hence in operator form, the differential equation reduces to ($D^2 + n^2$) $y = n^2a$

The solution of the above equation would consist of complementary solution andparticular solution, therefore

y_{gen} = A cos(nx) + sin(nx) + P. I

where

P.I = the P.I is a particular value of y which satisfies the differential equationHence

$y_{P.I}$ = a

Therefore the complete solution becomes Y =A

cos(nx) + B sin(nx) + a

Now imposing the boundary conditions to evaluate the constants A and B

(i) at x = 0; y = 0 This

yields A = -a

(ii) at x = 0; dy/dx = 0 This

yields B = 0 Hence

y = a cos(nx) + a Futher, atx =

L; y = a

Therefore a = - a cos(nx) + a   or 0 = cos(nL)

Now the fundamental mode of buckling in this case would be

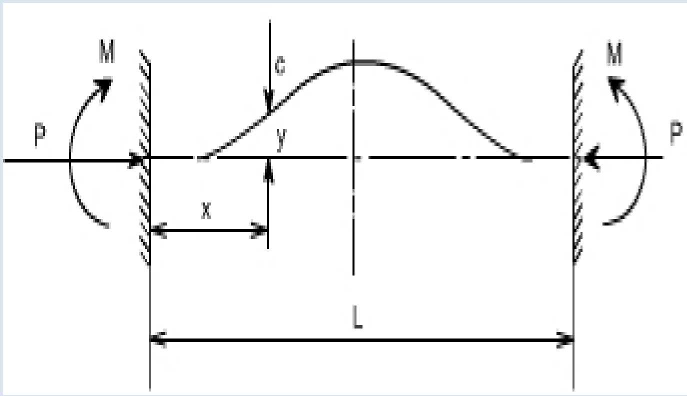
$$nL = \frac{\pi}{2}$$

$$\sqrt{\frac{P}{EI}}\,L = \frac{\pi}{2}, \text{Therefore, the Euler's crippling load is given as}$$

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3

Strut with fixed ends:



Due to the fixed end supports bending moment would also appears at the supports,since this is the property of the support.

Bending Moment at point C = M – P.y

$$EI\frac{d^2y}{dx^2} = M - Py$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$$

$n^2 = \frac{P}{EI}$, Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\,integral}$$

$$y|_{P.I} = \frac{M}{n^2 EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\,Cosnx + A\,Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at x=0:y=0

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

A=0

Therefore,

$$y = -\frac{M}{P}\,Cos\,nx + \frac{M}{P}$$

$$y = \frac{M}{P}\left(1 - Cos\,nx\right)$$

Futher, it may be noted that at $x = L; y = 0$

$$\text{Then } 0 = \frac{M}{P}\left(1 - Cos\,nL\right)$$

Thus, either $\frac{M}{P} = 0$ or $\left(1 - Cos\,nL\right) = 0$

obviously, $\left(1 - Cos\,nL\right) = 0$

$\cos nL = 1$

Hence the least solution would be

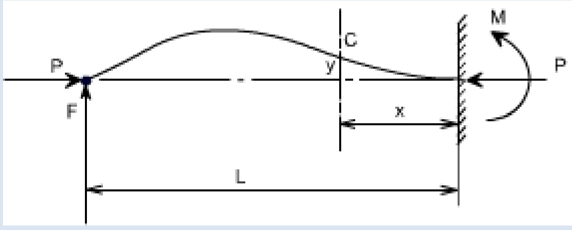
$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\,L = 2\pi$, Thus, the buckling load or crippling load is

Thus, $$P_e = \frac{4\pi^2.EI}{L^2}$$

Case 4

One end fixed, the other pinned



In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin. The moment of F about the built in end then balances the fixing moment.

With the origin at the built in end, the B,M at C is given as

$$EI\frac{d^2y}{dx^2} = -Py + F(L-x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L-x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L-x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L-x)$$

$$y_{particular} = \frac{F}{n^2EI}(L-x) \text{ or } y = \frac{F}{P}(L-x)$$

The full solution is therefore

$$y = A\,Cos\,mx + B\,Sin\,nx + \frac{F}{P}(L-x)$$

The boundry conditions relevants to the problem are at x=0;y=0

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x = 0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P}Cos\,nx + \frac{F}{nP}Sin\,nx + \frac{F}{P}(L-x)$$

$$y = \frac{F}{nP}\left[Sin\,nx - nL\,Cos\,nx + n(L-x)\right]$$

Also when x = L ; y = 0 Therefore

nL Cos nL = Sin nL or tan nL = nL

The lowest value of nL (neglecting zero) which satisfies this condition and whichtherefore produces the fundamental buckling condition is nL = 4.49radian

$$\text{or} \sqrt{\frac{P}{EI}}\, L = 4.49$$

$$\frac{P_e}{EI} L^2 = 20.2$$

$$P_e = \frac{2.05\pi^2\, EI}{L^2}$$

Equivalent Strut Length:

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.

$$\text{i.e. } P_e = \frac{\pi^2 EI}{L^2}$$

Where L is the equivalent length of the strut and can be related to the actual length of the strut depending on the end conditions.

The equivalent length is found to be the length of a simple bow(half sine wave) in each of the strut deflection curves shown. The buckling load for each end condition shown is then readily obtained. The use of equivalent length is not restricted to the Euler's theory and it will be used in other derivations later.

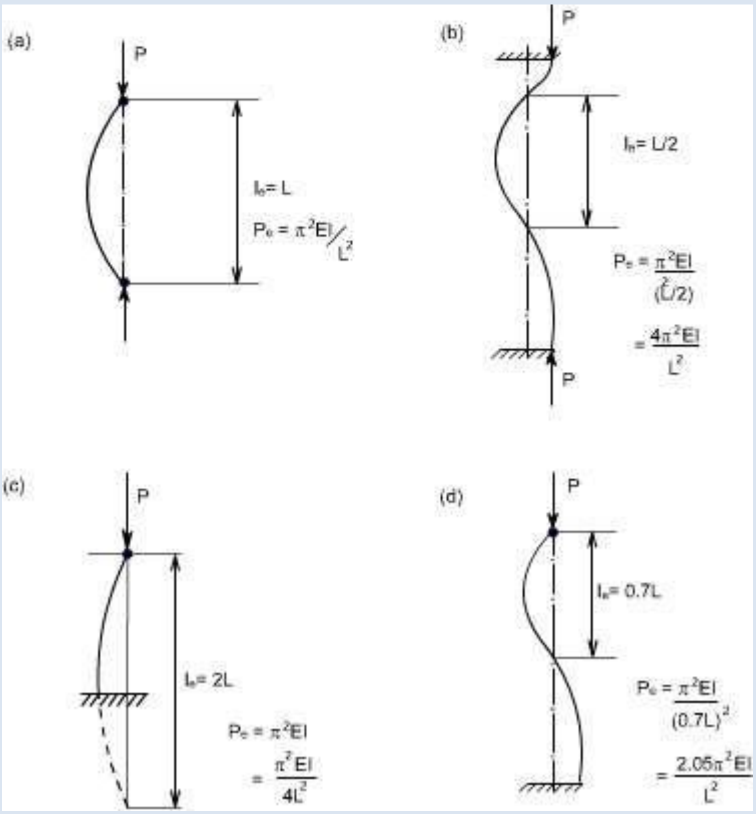
The critical load for columns with other end conditions can be expressed in terms of the critical load for a hinged column, which is taken as a fundamental case.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the freebody diagram would indicates that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we have considered so far are:

A. Both ends pinned C .One end fixed, other free

B. Both ends fixed D. One end fixed and other pinned



Limitations of Euler's Theory :

In practice the ideal conditions are never [i.e. the strut is initially straight and the end load being applied axially through centroid] reached. There is always some eccentricity and initial curvature present. These factors needs to be accommodated in the required formula's.

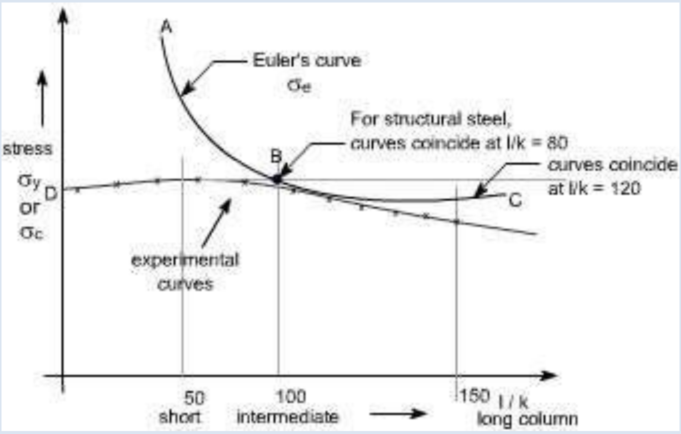
It is realized that, due to the above mentioned imperfections the strut will suffer a deflection which increases with load and consequently a bending moment is introduced which causes failure before the Euler's load is reached. Infact failure is by stress rather than by buckling and the deviation from the Euler value is more marked as the slenderness-ratio l/k is reduced. For values of l/k < 120 approx, the error in applying the Euler theory is too great to allow of its use. The stress to cause buckling from the Euler formula for the pin ended strut is

$$\text{Euler's stress, } \sigma_e = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$$

$$\text{But, } I = Ak^2$$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

A plot of $_e$ versus l / k ratio is shown by the curve ABC.



Allowing for the imperfections of loading and strut, actual values at failure must lie within and below line CBD.

Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio i.e.l/k=40 to l/k=100.

(a) **Straight – line formula :**

The permissible load is given by the formulae

$$P = \sigma_y A\left[1 - n\left(\frac{l}{k}\right)\right]$$

Where the value of index 'n' depends on the material used and the end conditions.

(b) **Johnson parabolic formula:** The Johnson parabolic formulae is defined as

$$P = \sigma_y A\left[1 - b\left(\frac{l}{k}\right)^2\right]$$

where the value of index 'b' depends on the end conditions.

(c) **Rankine Gordon Formula:**

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

Where P_e = Euler crippling load

P_c = Crushing load or Yield point load in Compression P_R =

Actual load to cause failure or Rankine load

Since the Rankine formulae is a combination of the Euler and crushing load for astrut.

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

For a very short strut P_e is very large hence 1/ P_e would be large so that 1/ P_e can beneglected. Thus $P_R = P_c$, for very large struts, P_e is very small so 1/ P_e would be large and 1/ P_c can be neglected ,hence $P_R = P_e$

The Rankine formulae is therefore valid for extreme values of 1/k.It is also found to be fairly accurate for the intermediate values in the range under consideration. Thus rewriting the formula in terms of stresses, we have

$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

$$\frac{1}{\sigma} = \frac{1}{\sigma_e} + \frac{1}{\sigma_y}$$

$$\frac{1}{\sigma} = \frac{\sigma_e + \sigma_y}{\sigma_e . \sigma_y}$$

$$\sigma = \frac{\sigma_e . \sigma_y}{\sigma_e + \sigma_y} = \frac{\sigma_y}{1 + \frac{\sigma_y}{\sigma_e}}$$

For struts with both ends pinned

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E}\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + a\left(\frac{l}{k}\right)^2} \qquad a = \frac{\sigma_y}{\pi^2 EI}$$

Where and the value of ‘a' is found by conducting experiments on various materials. Theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end conditions.